ON EMBEDDING THEOREMS FOR BESOV SPACES WITH MIXED NORM

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Abstract

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ON EMBEDDING THEOREMS FOR BESOV SPACES WITH MIXED NORM

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Embedding theorems for the spaces $W^l_{(p)}$ (l integer), $H^l_{(p)}$, $S^r_{(p)}H$, $L^r_{(p),(\theta)}$ with mixed norm have been considered in a number of works ($^{5-15}$). In the present note embedding theorems are given for the spaces $B^l_{(p)}(E^n)$ with mixed norm. Weighted spaces $B^l_{(p),\alpha}(\overset{+}{E}{}^n)$ are also considered. Let E^n be the n-dimensional real Euclidean space of points $\mathbf{x} = (x_1, \dots, x_n)$; $\overset{+}{E}{}^n = \{\mathbf{x} : x_n > 0\}$; $E^n = E^{n_1} \times \dots \times E^{n_k} = E^{m_1} \times \dots \times E^{m_\tau}$; $\overset{+}{E}{}^n = \widetilde{E}^{n_1} \times \dots \times \widetilde{E}^{n_k} = \widetilde{E}^{m_1} \times \dots \times \widetilde{E}^{m_\tau}$, and the decompositions into E^{n_i} and E^{m_j} ($\widetilde{E}^{n_i}$ and $\widetilde{E}^{m_j}$) ($i = 1, \dots, k$; $j = 1, \dots, \tau$) do not depend on one another; the vectors of the subspaces E^{n_i} and E^{m_j} will be denoted respectively by $\mathbf{x}_{n_i} = (x_{i1}, \dots, x_{in_i})$ and $\mathbf{x}_{m_j} = (x^{(j)}_1, \dots, x^{(j)}_{m_j})$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $\mathbf{p} = (p_1, \dots, p_k)$, $\mathbf{q} = (q_1, \dots, q_\tau)$, $\mathbf{l} = (l_1, \dots, l_n)$, $\mathbf{r} = (r_1, \dots, r_n)$; $\nu_i, \bar{l}_i, \bar{r}_i$ ($i = 1, \dots, n$) are nonnegative integers, $\alpha \geq 0$.

$\square^{\varkappa_{n_i}}_h(\mathbf{x}_{n_i})$ is an n_i-dimensional parallelepiped in E^{n_i} with vertex at the point $\mathbf{x}_{n_i}$ and edges $h^{\varkappa_{i1}}, \dots, h^{\varkappa_{in_i}}$ ($i = 1, \dots, k$).

Let $f(y)$ be a smooth finite function in E^n and

$$\|f\|_{\mathcal{L}^{l_i}_{(p),\alpha}(\overset{+}{E}{}^n)} = \left(\int_0^\infty \frac{dt}{t^{1+p_k\alpha_i}} \left\| y_n^{\alpha/p_1} \Delta_i^2(t) D_i^{\bar{l}_i} f(y) \right\|^{p_k}_{L_{(p)}(\overset{+}{E}{}^n)}\right)^{1/p_k} < \infty \qquad (1)$$

(where $L_{(p)}(\overset{+}{E}{}^n)$ is the space with mixed norm (see (8)));

$$\|f\|_{B^l_{(p),\alpha}(\overset{+}{E}{}^n)} = \|f\|_{L_{(p)}(\overset{+}{E}{}^n)} + \sum_{i=1}^n \|f\|_{\mathcal{L}^{l_i}_{(p),\alpha}(\overset{+}{E}{}^n)}. \qquad (2)$$

Putting in (2) $\alpha = 0$ and replacing $\overset{+}{E}{}^n$ by E^n, we obtain

$$\|f\|_{B^l_{(p)}(E^n)} = \|f\|_{L_{(p)}(E^n)} + \sum_{i=1}^{n} \|f\|_{\mathcal{L}^{l_i}_{(p)}(E^n)}. \tag{3}$$

By the spaces $B^l_{(p)}(E^n)$ and $B^l_{(p),\alpha}(\overset{+}{E}{}^n)$ we shall mean the closures of the set of smooth finite functions in the norms (2) and (3).

We formulate the main results.

Theorem 1. If $1 < p_i \le q_i < \infty$ $(i = 1, \dots, k;\ j = 1, \dots, \tau)$; $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \le 1$ $(i = 1, \dots, n)$;

$$1 - \sum_{i=1}^{n} \frac{1}{l_i}(1 + \nu_i) + \sum_{i=1}^{k} \frac{1}{p_i} \sum_{j=1}^{n_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \frac{1}{l_j^{(i)}} > 0,$$

$f \in B^l_{(p)}(E^n)$, then $D_1^{\nu_1} \cdots D_n^{\nu_n} f \in L_{(q)}(E^n)$, and the inequality

$$\left\|D_1^{\nu_1} \cdots D_n^{\nu_n} f\right\|_{L_{(q)}(E^n)} \le C\|f\|_{B^l_{(p)}(E^n)} \tag{4}$$

holds.

Theorem 2. If $1 < p_i \le q_j < \infty$ $(i = 1, \dots, k;\ j = 1, \dots, \tau)$; $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \le 1$, $r_i = \bar{r}_i + \beta_i$, $0 < \beta_i \le 1$ $(i = 1, \dots, n)$;

$$1 - \sum_{i=1}^{n} \frac{1}{l_i} + \sum_{i=1}^{k} \frac{1}{p_i'} \sum_{j=1}^{n_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_j} \frac{1}{l_j^{(i)}} > \frac{r_s}{l_s} \qquad (s = 1, \dots, n),$$

$f \in B^l_{(\mathbf{p})}(E^n)$, then $f \in B^r_{(\mathbf{q})}(E^n)$ and the inequality

$$\|f\|_{B^r_{(\mathbf{q})}(E^n)} \le C\|f\|_{B^l_{(\mathbf{p})}(E^n)} \tag{5}$$

holds.

Theorem 3. If $1 < p_i \le q_j < \infty$, $0 \le \alpha < p_1/\max_i p_i'$ $(i = 1, \dots, k;\ j = 1, \dots, \tau)$; $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \le 1$ $(i = 1, \dots, n)$;

$$1 - \sum_{i=1}^{n} \frac{1}{l_i}(1 + \nu_i) - \frac{\alpha}{p_1 l_n} + \sum_{i=1}^{k} \frac{1}{p_i'} \sum_{j=1}^{\tilde{n}_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{\tilde{m}_i} \frac{1}{l_j^{(i)}} > 0,$$

$f \in B^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$, then $D_1^{\nu_1} \cdots D_n^{\nu_n} f \in L_{(\mathbf{q})}(\overset{+}{E}{}^n)$ and the inequality

$$\|D_1^{\nu_1}\cdots D_n^{\nu_n} f\|_{L_{(\mathbf{q})}(\overset{+}{E}{}^n)} \le C\|f\|_{B^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)} \tag{6}$$

holds.

Theorem 4. If $1 < p_i \le q_j < \infty$, $\quad 0 \le \alpha < p_1/\max_i p_i'$ $(i = 1,\dots,k;\ j = 1,\dots,\tau)$; $l_i = \bar{l}_i + \alpha_i$, $\quad 0 < \alpha_i \le 1$, $\quad r_i = \bar{r}_i + \beta_i$, $0 < \beta_i \le 1$ $(i = 1,\dots,n)$;

$$1 - \sum_{i=1}^{n}\frac{1}{l_i} - \frac{\alpha}{p_1 l_n} + \sum_{i=1}^{k}\frac{1}{p_i'}\sum_{j=1}^{\tilde{n}_i}\frac{1}{l_{ij}} + \sum_{i=1}^{\tau}\frac{1}{q_i}\sum_{j=1}^{\tilde{m}_i}\frac{1}{l_j^{(i)}} > \frac{r_s}{l_s} \qquad (s = 1,\dots,n);$$

$f \in B^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$, then $f \in B^r_{(\mathbf{q})}(\overset{+}{E}{}^n)$ and the inequality

$$\|f\|_{B^r_{(\mathbf{q})}(\overset{+}{E}{}^n)} \le C\|f\|_{B^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)} \tag{7}$$

holds.

In inequalities (4)–(7) the constant C does not depend on f.

Let us indicate the scheme of proof, for example, of Theorem 2. Consider the norm $\|f\|_{\mathcal{L}^{r_s}_{(\mathbf{q})}(E^n)}$: its estimate, in view of Theorem 1, reduces to estimating the expression

$$N = \left(\int_0^{h^{\chi_s}} \frac{dz}{z^{1+q_\tau\beta_s}}\|\Delta_s^2(z)D_s^{\bar{r}_s}f(x)\|^{q_\tau}_{L_{(\mathbf{q})}(E^n)}\right)^{1/q_\tau}, \qquad \text{where } \chi_s = \frac{1}{l_s} \quad (s = 1,\dots,n).$$

Using the integral identity of V. P. Il’ in (^{3}), and making the necessary transformations and estimates, we obtain

$$\begin{aligned}
N \le C &\left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+(\beta_s-2)q_\tau}} h^{(-\delta-\bar{r}_s\chi_s-2\chi_s)}\|f\|^{q_\tau}_{L_{(\mathbf{p})}(E^n)}\right]^{1/q_\tau} \\
&+ C\sum_{i=1}^{n}\left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+\beta_s q_\tau}} \times \left\|\int_0^{z^{1/\chi_s}} \frac{dv}{v^{1+\lambda_i-\varepsilon+\gamma_i\chi_i+r_s\chi_s}}\int_0^{v^{\chi_i}} t^{\gamma_i-\left(\frac{1}{p_k}+\alpha_i\right)}\,dt\int_x^{x+v^k}\Delta_i^2\left(\frac{t}{2}\right)D_i^{\bar{l}_i}f(y)\,dy\right\|^{q_\tau}_{L_{(\mathbf{q})}(E^n} \right. \\
&+ C\sum_{i=1}^{n}\left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+(\beta_s-2)q_\tau}}\left\|\int_{z^{1/\chi_s}}^{h} \frac{dv}{v^{1+\lambda_i-\varepsilon+\bar{r}_s\chi_s+2\chi_s+\gamma_i\chi_i}}\int_0^{v^{\chi_i}} t^{\gamma_i-(1/p_k+\alpha_i)}\,dt\right.\right. \\
&\qquad \left.\left.\times\int_x^{x+v^k}\Delta_i^2\left(\frac{t}{2}\right)D_i^{\bar{l}_i}f(y)\,dy\right\|^{q_\tau}_{L_{(\mathbf{q})}(E^n)}\right]^{1/q_\tau} = N_1 + N_2 + N_3,
\end{aligned}$$

where

$$\delta = \sum_{i=1}^{n} \chi_i - \sum_{i=1}^{k} \frac{1}{p_k} \sum_{j=1}^{n_i} \chi_{ij} - \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \chi_j^{(i)},$$

$$0 < \gamma_i \leq \frac{1}{p_k} + \alpha_i, \qquad \lambda_i = \frac{\chi_i}{p_k} + \sum_{j=1}^{n} \chi_j - \delta, \qquad \varepsilon = 1 - \delta.$$

Let us estimate one of the expressions, for example N_2 (N_1, N_3 are estimated analogously). For this we choose γ_i so that $\varepsilon - r_s\chi_s > \gamma_i\chi_i$ $(i = 1, \dots, n;\ s = 1, \dots, n)$; then, after elementary estimates, we shall have

$$N_2 \leq c h^{\varepsilon - r_s\chi_s} \sum_{i=1}^{n} \Bigg[\int_0^{h^{\chi_s}} \frac{dz}{1 + \frac{q_\tau}{\chi_s}(\gamma_i\chi_i - \mu)} \Bigg\| \Bigg(\int_0^{z^{1/\chi_s}} \frac{dv}{1 + p_k \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \chi_j^{(i)} + \mu p_k}$$

$$\times \int_0^{v^{\chi_i}} t^{\chi_i p_k - (1 + p_k\alpha_i)}\, dt \left\| \Delta_i^2\left(\frac{t}{2}\right) \bar{D}_i^{l_i} f(y) \right\|^{p_k}_{L^{(p)}\left[\square_v x_{n_1}(x_{n_1}), \dots, \square_v x_{n_k}(x_{n_k})\right]} \Bigg)^{1/p_k} \Bigg\|^{q_\tau}_{L^{(q)}(E^n)} \Bigg]^{1/q_\tau} \leq$$

(where μ is an arbitrary positive number); using the generalized Minkowski inequality and lemma (1) of A. Kh. Gudiev [5] the required number of times, we shall have

$$\leq c h^{\varepsilon - r_s\chi_s} \sum_{i=1}^{n} \Bigg[\int_0^{h^{\chi_s}} \frac{dz}{1 + \frac{q_\tau}{\chi_s}(\gamma_i\chi_i - \mu)} \Bigg(\int_0^{z^{1/\chi_s}} \frac{dv}{v^{1 + \mu p_k}} \int_0^{v^{\chi_i}} t^{\chi_i p_k - (1 + p_k\alpha_i)}\, dt$$

$$\times \left\| \Delta_i^2\left(\frac{t}{2}\right) \bar{D}_i^{l_i} f(y) \right\|^{p_k}_{L^{(p)}(E^n)} \Bigg)^{q_\tau} \Bigg]^{1/q_\tau} \leq$$

changing the order of integration; choosing μ so that $\gamma_i\chi_i - \mu > 0$, we obtain

$$\leq c h^{\varepsilon - r_s\chi_s} \sum_{i=1}^{n} \|f\|_{L^{l_i}_{(p)}(E^n)}.$$

From the estimates obtained, for $h = 1$ we obtain inequality (5).

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Note: Figure translations are in progress. See original paper for figures.

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