ON THE INDEX OF A PSEUDODIFFERENTIAL OPERATOR WITH A FINITE GROUP OF SHIFTS

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Abstract

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MATHEMATICS

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ON THE INDEX OF A PSEUDODIFFERENTIAL OPERATOR WITH A FINITE GROUP OF SHIFTS

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Let G be a finite group of order N; let X be a connected G-manifold and let the action of the group G on X be effective. In the space $C_\infty(X)$, by the formula

$$T_g u(x) = u(g^{-1}x), \qquad g \in G, \qquad x \in X, \qquad u \in C_\infty(X),$$

we define a representation T_g of the group G.

By a pseudodifferential operator of order r with a finite group of shifts we shall mean an operator A acting in $C_\infty(X)$ by the formula

$$Au = \sum_{g \in G} A_g T_g u,$$

where the A_g are pseudodifferential operators of order r. The operator A belongs to the operators with a deviating argument. In the present paper conditions for the Noether property of the operator A are indicated and a formula for its index is given.

Let $\widetilde{A}$ be the operator acting in the space $C_\infty(X \times G)$ by the formula

$$\widetilde{A}u(x, g') = \sum_{g \in G} T_{g'} A_{gT_{g'}}^{-1} u(x, g'g).$$

If functions from $C_\infty(X \times G)$ are regarded as vector-functions on X, then the operator $\widetilde{A}$ is a G-invariant pseudodifferential operator ([1]).

Theorem 1. *In order that the extension of the operator A be a Noether operator from $H_l(X)$ into $H_{l-r}(X)$, it is necessary and sufficient that the operator $\widetilde{A}$ be elliptic.*

Proof. In the space $C_\infty(X \times G)$ define a representation $\widetilde{T}_g$ of the group G

$$\widetilde{T}_{g''}u(x,g') = u(g''^{-1}x, g''^{-1}g').$$

The space $C_\infty(X \times G)$ decomposes into a direct sum of a finite number of invariant subspaces M^i $(i = 1, \dots, k)$ such that in M^i there acts a representation which is a multiple of the irreducible representation D^i of the group G. The operator $\widetilde{A}$ commutes with all operators $\widetilde{T}_{g''}$ and, consequently, the subspaces M^i are invariant with respect to the operator $\widetilde{A}$, i.e. the operator $\widetilde{A}$ decomposes into the direct sum of operators A_i, $i = 1, \dots, k$. Let M^1 and A_1 be the subspace and the operator corresponding to the identity representation of the group G. The mapping $p : M^1 \to C_\infty(X)$, acting by the formula

$$pu(x) = u(x, e),$$

is an isomorphism and carries the operator A_1 into A. The remaining operators A_i can also be realized in $C_\infty^{n_i}(X)$ as pseudodifferential operators with shift (n_i is the dimension of the representation D^i).

Let now $\widetilde{A}$ be an elliptic operator. Then it can be extended to a Noetherian operator acting from $H_l^N(X)$ to $H_{l-r}^N(X)$, and consequently the operator A is also extended to a Noetherian operator from $H_l(X)$ to $H_{l-r}(X)$. Suppose that $\widetilde{A}$ is not an elliptic operator. Then one can construct a function $u_\lambda \in M^1$ such that, as $\lambda \to \infty$, the a priori estimate is not satisfied for u_λ, which contradicts the Noetherian property of the operator A. The theorem is proved.

In work [1] the analytic index of a G-invariant operator is defined as an element of the ring $R(G)$ of characters of representations of the group G:

$$\operatorname{index} B = [\operatorname{Ker} B] - [\operatorname{Coker} B] \in R(G),$$

where $[\operatorname{Ker} B]$, $[\operatorname{Coker} B]$ are the characters of the representations induced by the action of the group G in $\operatorname{Ker} B$ and $\operatorname{Coker} B$. The value of $\operatorname{index} B$ at an element g is called the Lefschetz number and is denoted by $L(g, B)$. In [1–3] a method is indicated for computing the index for pseudodifferential operators.

Theorem 2. *The index of a pseudodifferential operator A with a finite group of shifts is expressed by the formula*

$$\operatorname{ind} A = \frac{1}{N} \sum_{g \in G} L(g, \widetilde{A}). \tag{1}$$

Proof. From the decomposition of the operator $\widetilde{A}$ we obtain

$$\operatorname{index} \widetilde{A} = \sum_{i=1}^{k} \operatorname{index} A_i, \tag{2}$$

and since in M^i there acts a representation that is a multiple of the irreducible representation D^i, we have

$$\text{index}\, A_i = \frac{\text{ind}\, A_i}{n_i}\chi_i, \tag{3}$$

where χ_i is the character of the irreducible representation D^i, $\text{ind}\, A_i = \dim \text{Ker}\, A_i - \dim \text{Coker}\, A_i$. By virtue of the orthogonality of the characters of irreducible representations, from formulas (2) and (3) we obtain

$$\text{ind}\, A_i = \frac{n_i}{N} \sum_{g\in G} L(g, \widetilde{A})\overline{\chi_i}(g). \tag{4}$$

From (4), for $i = 1$, we obtain the assertion of the theorem.

Theorem 3. *If every transformation $g \in G$, $g \neq e$, has a finite number of fixed points, then*

$$\text{ind}\, A = \frac{1}{N}\, \text{ind}\, \widetilde{A}. \tag{5}$$

Proof. If g has a finite number of fixed points, then in work [2] $L(g, \widetilde{A})$ is expressed in terms of the trace of the operator $\widetilde{T}_g$ in the fiber over a fixed point. Since in the fiber over a fixed point $\widetilde{T}_g$ acts as the regular representation, $\text{Tr}\, T_g = 0$, $L(g, \widetilde{A}) = 0$, and formula (1) gives (5).

The results of the article are easily transferred to operators in sections of vector bundles. In the case of one-dimensional singular integral operators and the cyclic group G, formula (5) was obtained by other methods in works [4,5].

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Note: Figure translations are in progress. See original paper for figures.