CLASSIFICATION AND CHARACTERISTIC PROPERTIES OF CARLEMAN OPERATORS

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1°. Classification of Carleman operators.
Let (X, S, μ) be a space with a completely σ-finite measure (([1]), p. 77) and let $K(s,t)$ be a $(\mu \times \mu)$-measurable function defined on $X \times X$. Below we consider integral operators with kernels satisfying the following (not necessarily all*) conditions:

(I) T. Carleman' s condition ([2], [3])

$$\int_X |K(s,t)|^2 \, d\mu(t) < \infty$$

for μ-almost all (a.e.) $s \in X$.

(II) $K(s,t) = \overline{K(t,s)}$ for $(\mu \times \mu)$-a.e. $(s,t) \in X \times X$.

(III) N. I. Akhiezer' s condition ([3], [4]): there exists a μ-measurable, μ-a.e. finite, nonnegative function $P(s)$, defined on X, such that

$$|K(s,t)| \leq P(s)P(t)$$

for $(\mu \times \mu)$-a.e. $(s,t) \in X \times X$.

Functions satisfying conditions (I), (II) are called Carleman kernels ([2], [3]) (abbreviated (C)-kernels). Functions satisfying condition (I) are called semi-Carleman kernels ([5]) ((SC)-kernels). Functions satisfying condition (III) are called B-kernels ([4]). A (C)-kernel satisfying condition (III) will be called a (BC)-kernel. An (SC)-kernel satisfying condition (III) will be called a (BSC)-kernel.

A densely defined integral operator acting in $L_2(X, S, \mu)$,

$$Lf = \int_X K(s,t) f(t) \, d\mu(t), \qquad f \in D_L, \tag{1}$$

with $(\mu \times \mu)$-measurable kernel $K(s,t)$, will be called: a (C)-operator, if $K(s,t)$ is a (C)-kernel; an (SC)-operator, if $K(s,t)$ is an (SC)-kernel; a (BC)-operator, if $K(s,t)$ is a (BC)-kernel; a (BSC)-operator, if $K(s,t)$ is a (BSC)-kernel.

As in ([6]), a densely defined linear operator T acting in $L_2(X,S,\mu)$ will be called an operator of type (C) if it is unitarily equivalent to a (C)-operator. Operators of types (SC), (BC), and (BSC) are defined analogously.

Finally, following ([6]), we shall call an operator T a strong (C)-operator if, for every unitary operator U in $L_2(X,S,\mu)$, the operator UTU^{-1} is a (C)-operator. Strong (SC)-, strong (BC)-, and strong (BSC)-operators are defined analogously.

In the present paper, for each of the classes (C), (SC), (BC), (BSC), the following three problems are considered**:

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* But necessarily Carleman' s condition (I). We call such operators Carleman operators.

** The first two problems for the class (C) were posed in ([7]). The third problem (for the class (SC)) was posed in ([6]) and studied in ([6, 9]).

Find necessary and sufficient conditions under which the operator T is: 1) an operator of the given class; 2) an operator of the given type; 3) a strong operator of the given class.

The paper gives solutions of each of the three problems for all four classes. The results obtained are, in a certain sense, final, with the exception of one case (BSC)–2)–the problem is solved only for normal operators.

Everywhere below in the paper it is assumed that (X,S,μ) is a separable space (([1]), p. 165) with a completely σ-finite measure (([1]), p. 77), and that T is a densely defined linear operator in $L_2(X,S,\mu)$.

Theorem 1. 1) The operator T is an (SC)-operator if and only if T has a majorant, i.e., if there exists a μ-measurable, μ-a.e. finite nonnegative function $\Lambda(s)$ defined on X such that for all $f \in D_T$

$$|(Tf)(s)| \le \Lambda(s)\|f\| \quad \text{for } \mu\text{-a.e. } s \in X. \tag{2}$$

2) Suppose that the measure μ is not purely atomic*. The operator T is an operator of type (SC) if and only if the adjoint operator T^* is densely defined and the limiting spectrum of T^* contains 0.

3) Suppose that the measure μ is not purely atomic. The operator T is a strong (SC)-operator if and only if its closure is a Hilbert–Schmidt operator.

The second assertion of Theorem 1, in the case where T is a normal operator, $X = (a,b)$, and μ is Lebesgue measure, coincides with Theorem 1 of paper ([6]).

The third assertion of Theorem 1, in the case where $X = (a, b)$ and μ is Lebesgue measure, is a strengthening of Theorem 4 of paper ([6]) and Theorem 3 of paper ([9]).

Remark. Condition (2) of Theorem 1 is equivalent to the following conditions (([3]), p. 463):

$$D_{T^*} \supset [L_2]_\Lambda = \{f : \ f \in L_2(X, S, \mu), \quad \|f\|_\Lambda = \int \Lambda(s)|f(s)|\, d\mu(s) < \infty\} \quad (3)$$

$$\|T^* f\| \le \|f\|_\Lambda \quad \text{for all } f \in [L_2]_\Lambda.$$

Theorem 2. 1) The operator T is a (C)-operator if and only if the operator T has a majorant $\Lambda(s)$ such that the inclusion (3) holds and**

$$(T^* f, g) = (f, T^* g) \quad \text{for all } f, g \in [L_2]_\Lambda. \quad (4)$$

2) Suppose that the measure μ is not purely atomic. The operator T is an operator of type (C) if and only if there exists a symmetric operator A such that $A \subseteq T^*$ and the limiting spectrum of the operator A contains 0.

3) Suppose that the measure μ is not purely atomic. The operator T is a strong (C)-operator if and only if its closure is a self-adjoint Hilbert-Schmidt operator.

The first assertions of Theorems 1 and 2 for the case $X = \Omega \subseteq R_n$, μ Lebesgue measure, were proved in ([8]). The second assertion of Theorem 2 is a generalization of a well-known theorem of J. von Neumann (([7]); ([3]), p. 467).

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* We shall say that the measure μ is not purely atomic if in X there exists a set E of finite nonzero μ-measure such that, for any atom τ, $\mu(E \cap \tau) = 0$. It can be shown that if μ is purely atomic, then any bounded operator in $L_2(X, S, \mu)$ is a strong (BSC) and, consequently, a strong (SC)-operator.

** Condition (4) ensures the Hermiticity of the kernel and was first considered by N. I. Akhiezer (([4]), p. 129).

Theorem 3. 1) The operator T is a (BSC)-operator if and only if the operator T has a majorant $\Lambda(s)$ such that $D_{T^*} \supset [L_2]_\Lambda$ and, for all $f \in [L_2]_\Lambda$,

$$|(T^* f)(t)| \le \Lambda(t)\|f\|_\Lambda \quad \text{for } \mu\text{-a.e. } t \in X, \quad (5)$$

where

$$\|f\|_\Lambda = \int_X \Lambda(s)|f(s)|\, d\mu(s).$$

2) Suppose that the measure μ is not purely atomic. A normal operator N is an operator of type (BSC) if and only if the residual spectrum of the adjoint operator N^* contains 0.

3) Suppose that the measure μ is not purely atomic. The operator T is a strong (BSC)-operator if and only if its closure is a nuclear operator.

Remark. Condition (5) of Theorem 3.1) is equivalent to the condition

$$|(T^* f, g)| \leq \|f\|_\Lambda \|g\|_\Lambda$$

for all $f, g \in [L_2]_\Lambda$, introduced by N. I. Akhiezer in the study of symmetric (BC)-operators (([4]), p. 129).

Theorem 4. 1) The operator T is a (BC)-operator if and only if the conditions of Theorem 3.1) and condition (4) of Theorem 2.1) are satisfied.

2) Suppose that the measure μ is not purely atomic. An operator is an operator of type (BC) if and only if there exists a symmetric operator A such that $A \subseteq T^*$ and the residual spectrum of the operator A contains 0.

3) Suppose that the measure μ is not purely atomic. The operator T is a strong (BC)-operator if and only if its closure is a self-adjoint nuclear operator.

3°. Some generalizations of Theorem 1.

A. A densely defined linear integral operator L in $L_2(X, S, \mu)$, taking values in $L_2(X_0, S_0, \mu_0)$, will be called a Carleman operator if its kernel $K(s,t)$ is $(\mu_0 \times \mu)$-measurable and

$$\int_X |K(s,t)|^2 \, d\mu(t) < \infty \quad \text{for } \mu_0\text{-a.e. } s \in X_0.$$

Theorem 5. 1) The operator τ is a Carleman operator if and only if τ has a majorant.

2) Suppose that (X_0, S_0, μ_0) is a separable space and that the measure μ_0 is not purely atomic. In order that there exist in $L_2(X_0, S_0, \mu_0)$ a unitary operator U such that $U\tau$ is a Carleman operator, it is necessary and sufficient that the adjoint operator τ^* be densely defined and that in the domain D_{τ^*} there exist an orthonormal sequence $\{f_n\}$ such that

$$\lim_{n\to\infty} \|\tau^* f_n\| = 0.$$

3) Suppose that (X_0, S_0, μ_0) is a separable space and that the measure μ_0 is not purely atomic. The operator $U\tau$ is a Carleman operator for every unitary operator U in $L_2(X_0, S_0, \mu_0)$ if and only if the closure of the operator τ is a Hilbert–Schmidt operator.

B. A densely defined linear operator L in H, taking values in $L_2(H_0; X_0, S_0, \mu_0)$ (H, H_0 are separable Hilbert spaces; for the definition of $L_2(H_0; X_0, S_0, \mu_0)$, see ([10]), p. 103), will be called Carleman if there exists a strongly μ_0-measurable operator function $L(s)$ defined on X_0 (([10]), p. 88), taking values in the space of bounded linear operators acting from H to H_0, such that $(Lh)(s) = L(s)h$ for μ_0-a.e. $s \in X_0$ ($h \in D_L$). For such operators a theorem analogous to Theorem 5 is valid.

In conclusion, we note that the sets of full measure on which (1), (2), (5) are fulfilled depend on f.

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Note: Figure translations are in progress. See original paper for figures.