GEOMETRY OF SPACES OF CONSTANT CURVATURE AS A SPECIAL CASE OF THE THEORY OF PHYSICAL STRUCTURES

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Abstract

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Yu. I. KULAKOV

GEOMETRY OF SPACES OF CONSTANT CURVATURE AS A SPECIAL CASE OF THE THEORY OF PHYSICAL STRUCTURES

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As is known, there exist several variants of the axiomatic construction of Euclidean geometry. In the present article a certain new method is proposed for the axiomatic construction of locally Euclidean, locally symplectic, and locally non-Euclidean geometries of constant curvature. This method may be regarded as a special case of the general principle of phenomenological symmetry proposed in [1] for a uniform description of various physical theories of phenomenological type, such as, for example, mechanics, thermodynamics, special relativity, etc.

In the present particular case the principle of phenomenological symmetry is formulated as follows.

Let $\mathfrak{M} = \{i, k, \dots, l, \dots\}$ be a set of objects of arbitrary nature, called points, and let each ordered pair (i, k), $i \neq k$, be assigned a real number a_{ik}. Here it is assumed* that one of the following two conditions is satisfied:

I_s. For all pairs (i, k), $i \neq k$, $a_{ik} = a_{ki}$.

I_a. For all pairs (i, k), $i \neq k$, $a_{ik} = -a_{ki}$.

Let $\mathfrak{M}_n = \{i_1, i_2, \dots, i_n\}$ be an arbitrary n-element subset of the set $\mathfrak{M}$. It corresponds to a system $\mathfrak{A}_n$ of $\frac{1}{2}n(n-1)$ numbers:

$$\mathfrak{A}_n = \{a_{i_r i_s}\},$$

where r and s run through all possible values such that $1 \leq r < s \leq n$.

We shall regard the system $\mathfrak{A}_n$ as a point in the $\frac{1}{2}n(n-1)$-dimensional arithmetic space $R^{n(n-1)/2}$. The set of all points of the space $R^{n(n-1)/2}$ assigned in the indicated manner to all possible n-element subsets $\mathfrak{M}_n$ will be denoted by $\mathfrak{G}_n$.

We introduce the following basic condition:

II. The set $\mathfrak{G}_n$ is a $\frac{1}{2}n(n-1)-1$-dimensional differentiable manifold of class C_∞.

Condition II constitutes the main content of the principle of phenomenological symmetry as applied to the case under consideration. If on the set $\mathfrak{M}$ a function of pairs a_{ik} $(i \neq k)$ is given such that conditions I and II are satisfied, then we shall say that a physical structure of rank n is given on the set $\mathfrak{M}$.

We introduce the notion of equivalence of physical structures. Two physical structures of the same rank n, given by the functions a_{ik} and b_{ik}, will be called equivalent if there exists a strictly monotone function of one variable $\chi(x)$ such that for any i and k, $b_{ik} = \chi(a_{ik})$. (We note that, in describing experimental reality, the arbitrariness of choosing the function $\chi(x)$ may be interpreted as the arbitrariness of choosing the scale of a measuring instrument.)

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* It can be shown that the condition "either $a_{ik} = a_{ki}$, or $a_{ik} = -a_{ki}$" is too restrictive; it is sufficient to require that $a_{ik} = f(a_{ki})$, where $f(x)$ is an *a priori* unknown function.

The problem consists in finding, for each $n \geq 2$, a function a_{ik} satisfying the requirements formulated above.

As a preliminary analysis shows ([2]), the requirement of phenomenological symmetry is extremely rigid in the sense that the number of possible physical structures of a given rank n is quite small.

It can be shown ([3]) that for the known locally Euclidean geometry, locally non-Euclidean geometry of constant curvature, and locally symplectic geometry, the principle of phenomenological symmetry holds (n-dimensional geometries realize the corresponding physical structures of rank $n+2$, if by a_{ik} one understands the corresponding distance functions between the elements i and k); and the manifold $\mathfrak{G}_n$ itself is specified, up to equivalence, by the following equations:

$$\mathrm{I_s}. \quad a_{ik} = a_{ki}$$

1)

$$\begin{vmatrix} a & a_{ik} & a_{il} & \cdots & a_{im} \\ a_{ik} & a & a_{kl} & \cdots & a_{km} \\ a_{il} & a_{kl} & a & \cdots & a_{lm} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{im} & a_{km} & a_{lm} & \cdots & a \end{vmatrix} = 0$$

or, in parametric form,

$$a_{ik} = g^0_{\sigma\rho} x^\sigma(i) x^\rho(k), \qquad \sigma, \rho = 1, 2, \ldots, n-1,$$

where $x^\sigma(i)$ $(\sigma = 1, 2, \ldots, n-1)$ are $(n-1)$ real parameters belonging to the point i and related to one another by a single relation

$$g^0_{\rho\sigma} x^\rho(i) x^\sigma(i) = a,$$

a is an arbitrary constant;

2)

$$g^0_{\rho\sigma} = \begin{cases} 0, & \rho \neq \sigma, \\ \pm 1, & \rho = \sigma. \end{cases}$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & a_{ik} & a_{il} & \cdots & a_{im} \\ 1 & a_{ik} & 0 & a_{kl} & \cdots & a_{km} \\ 1 & a_{il} & a_{kl} & 0 & \cdots & a_{lm} \\ . & . & . & . & \cdots & . \\ 1 & a_{im} & a_{km} & a_{lm} & \cdots & 0 \end{vmatrix} = 0$$

or, in parametric form,

$$a_{ik} = g^0_{\mu\nu}(x^\mu(i) - x^\mu(k))(x^\nu(i) - x^\nu(k)), \qquad \mu, \nu = 1, 2, \dots, n-2,$$

where $x^\mu(i)$ ($\mu = 1, 2, \dots, n-2$) are $(n-2)$ arbitrary real parameters belonging to the point i.

$$\mathrm{I_a}. \quad a_{ik} = -a_{ki}.$$

1) $n = 2p - 1$,

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & a_{ik} & a_{il} & \cdots & a_{im} \\ -1 & -a_{ik} & 0 & a_{kl} & \cdots & a_{km} \\ -1 & -a_{il} & -a_{kl} & 0 & \cdots & a_{lm} \\ . & . & . & . & \cdots & . \\ -1 & -a_{im} & -a_{km} & -a_{lm} & \cdots & 0 \end{vmatrix} = 0$$

or in parametric form:

$$a_{ik} = x^0(i) - x^0(k) + x^\nu(i) y_\nu(k) + x^\nu(k) y_\nu(i), \qquad \nu = 1, 2, \dots, p-2,$$

where $x^0(i), x^\nu(i), y_\nu(i)$ ($\nu = 1, 2, \dots, p-2$) are $n-2$ arbitrary real parameters associated with the point i.

2) $n = 2p$

$$\begin{vmatrix} 0 & a_{ik} & a_{il} & \dots & a_{im} \\ -a_{ik} & 0 & a_{kl} & \dots & a_{km} \\ -a_{il} & -a_{kl} & 0 & \dots & a_{lm} \\ \dots & \dots & \dots & \dots & \dots \\ -a_{im} & -a_{km} & -a_{lm} & \dots & 0 \end{vmatrix} = 0$$

or in parametric form:

$$a_{ik} = x^\lambda(i) y_\lambda(k) - x^\lambda(k) y_\lambda(i), \qquad \lambda = 1, 2, \dots, p-1,$$

where $x^\lambda(i), y_\lambda(i)$ $(\lambda = 1, 2, \dots, p-1)$ are $n-2$ arbitrary real parameters associated with the point i.

It is easy to see that the manifold $\mathfrak{C}_n$ in the form $\mathrm{I}_s(1)$ corresponds to an $(n-2)$-dimensional space of constant (positive or negative) curvature (curvature $1/a$); the manifold $\mathfrak{C}_n$ in the form $\mathrm{I}_s(2)$ corresponds to an $(n-2)$-dimensional Euclidean (or pseudo-Euclidean) space; and $\mathfrak{C}_n$ in the forms $\mathrm{I}_a(1)$ and $\mathrm{I}_a(2)$ corresponds to an $(n-2)$-dimensional symplectic space.

Thus, all the manifolds $\mathfrak{C}_n$ listed above are special cases of manifolds realizing a physical structure of rank n. (These manifolds we shall call phenomenological spaces of dimension $n-2$.)

The question arises: are symplectic spaces and spaces of positive, zero, and negative curvature the only types of phenomenological spaces?

For the case $n = 3$, an exhaustive solution of the problem was obtained by G. G. Mikhailichenko, who showed that all phenomenological spaces of dimension 1 are equivalent to the Euclidean line.

As for the cases $n \geq 4$, the question of the existence of any new types of phenomenological spaces, distinct from Euclidean, symplectic, and constant-curvature spaces, remains open.

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Novosibirsk State
University

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3. Yu. I. Kulakov, *Elements of the theory of physical structures*, Novosibirsk Univ., 1969.

Note: Figure translations are in progress. See original paper for figures.