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MAPPINGS OF FAMILIES OF SETS

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Abstract

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MATHEMATICS

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MAPPINGS OF FAMILIES OF SETS

1. We are concerned with the following question, which was considered in the preceding article ([1]). Let A_n, A'_m be two affine spaces, T, T' the groups of their translations, and let M be a bounded set in A_n. Let f be a one-to-one mapping of A_n onto A'_m such that for every $t \in T$ there exists a $t' \in T'$ such that

$$ft(M) = t'f(M), \tag{1}$$

and conversely: for every $t' \in T'$ there exists a $t \in T$ for which (1) is satisfied. The question is what such mappings can be. In ([1]) an answer to this question was given for a broad class of sets M, but we were forced to omit the proofs. In the present communication brief proofs are given of more particular results, which include, however, the cases when M is a convex or an arbitrary smooth domain.

2. In ([1]) the following construction was given. Let O be a point in A_n. Putting $M = M_0$, we define the sets M_i inductively:

$$M_i = \bigcup t(M_{i-1}), \qquad O \in t(M_{i-1}). \tag{2}$$

These sets (for $i \geq 1$) possess the following properties.

(I). If M_{iX} is constructed about the point X in the same way as M_i is about O, then $M_{iX} = t(M_i)$, where t is the translation $O \to X$.

(II) O is a center of symmetry of M_i. The proof is obvious.

(III) Let M'_i $(i = 1, 2, ...)$ be the sets constructed in A'_m about the point $O' = f(O)$, starting from the set $M' = f(M)$, in the same way as the M_i are constructed from M. Then $M'_i = f(M_i)$, and in general, for $X' = f(X)$,

$$M'_{iX'} = f(M_{iX}). \tag{3}$$

This follows from property (1) of the mapping f. If t is the translation $O \to X$, and t' is the translation $O' = f(O) \to X' = f(X)$, then (3) is equivalent to

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$$f(M_{iX}) = ft(M_i) = t'f(M_{iX}) = M'_{iX'}, \qquad t'f(M_i) = ft(M_i). \tag{4}$$

Thus, for the set M_i, condition (1) is satisfied with the additional property that the center $t(M_i)$ is mapped to the center $t'f(M_i)$. Therefore, instead of the given M, one may consider any of the M_i.

3. We formulate the theorems to be proved. In Theorems 1-3 it is assumed that f is continuous, and therefore is a homeomorphism of A_n onto A'_m (so that $m = n$). Moreover, everywhere M is understood to mean one of the M_i, and $M' = f(M)$ is bounded, while $n \geq 2$. A point X of a set N will be called extreme from within if it is a limit point of its interior and is not contained in any simplex with vertices in N distinct from X. At such a point N has a supporting plane, and if it is tangent to the set N at such a point X, i.e. is a contingency of its boundary ∂N at X, then it is the unique supporting plane to N at the point X.

Theorem 1. *If M has an extreme-from-within point and at it a tangent supporting plane P_0, then $f(P_0)$ is an $(n-1)$-dimensional plane in A'_n. For every plane $P \parallel P_0$, $f(P)$ is a plane parallel to $f(P_0)$.*

It follows immediately from Theorem 1 that if M has $(n+1)$ tangent planes in general position at points that are extreme from within, then f is affine. But from it one also derives the stronger assertion that f is affine if there are only n such planes.

4. A convex body may have no tangent planes at any extreme point, and then Theorem 1 gives nothing. However, the following holds.

Theorem 2. *If M is a convex body, but not a cylinder, then f is affine.*

A cylinder C is the Cartesian product of a segment and a flat set. In general,

$$C = l_1 \times ... \times l_k \times N, \tag{5}$$

where the l_i are segments, and N is contained in an $(n-k)$-dimensional plane and is no longer decomposable into the product of a segment and a set of smaller dimension. In the extreme case $C = l_1 \times ... \times l_n$ is a parallelepiped, and then in (5) N is understood to be a point. To each l_j there corresponds a base plane P_j, spanned by the remaining segments l_i and the set N.

Let P be an $(n-1)$-dimensional plane in A_n and let l be a vector not parallel to P. By a shear d_{Pl} we mean such a homeomorphism of A_n onto itself which is a parallel translation on every plane parallel to P, and sends every segment equal and parallel to l into a segment again equal and parallel to l. The segments l_i in formula (5) may be regarded as vectors. Then it is clear that every shear $d = d_{l_iPi}$ (where P_i is the base plane) is equivalent to some translation of the cylinder C, i.e. $d(C) = t(C)$. Comparing this with (1), we see that every shear

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$d_{l_i P i}$ is a mapping f of A_n onto itself when $M = C$. At the same time, the following holds.

Theorem 3. *If M is a convex cylinder, then*

$$f = f_0 d_1 \dots d_k, \tag{6}$$

where f_0 is an affine mapping of A_n onto A'_n, and the d_i are arbitrary shears $d_{l_i P i}$. Moreover, they can be normalized so that $d_i(M) = M$, so that one obtains $f(M) = f_0(M)$, i.e. $f(M)$ is necessarily an affine image of M.

If the original M is convex, then every M_i is convex. The converse is, of course, false. The most important case is when M is the boundary of a convex body. Then M_1 will, as is easy to verify, be a convex body. Therefore, from Theorems 2, 3 it follows that if M is the boundary of a convex body H and H is not a cylinder, then f is affine; if, however, H is a cylinder, then (6) holds.

5. Concerning the continuity of f, we shall prove the following theorem.

Theorem 4. *If M is open or closed with interior points and f^{-1} is bounded, then f is continuous.*

6. We shall prove Theorem 1. Let M have an extreme interior point B and, at it, a tangent plane P_0. By central symmetry of M there is a symmetric extreme point $\widetilde{B}_0$ and, at it, a tangent plane $\widetilde{P}_0 \parallel P_0$. From the construction (2) of the sets M_i it is easy to conclude that the same is true for each M_i: it has a pair of symmetric extreme interior points $B_i, \widetilde{B}_i$ with tangent planes $P_i, \widetilde{P}_i$, parallel to P_0.

Take a plane $P \parallel P_0$ and a point $X \in P$. Subject each M_i to two translations: $B_i \to X$ and $\widetilde{B}_i \to X$. We obtain sets $t(M_i)$, $\tilde{t}(M_i)$ with common point X and common tangent supporting plane P. Here $t(M_i)$ and $\tilde{t}(M_i)$ lie on different sides of P and have no other common points except X, otherwise X would not be extreme for them.

The sets $t(M_i)$ and $\tilde{t}(M_i)$ are mutually symmetric with respect to X, since they are centrally symmetric, situated in parallel fashion, and have only one common point X. Therefore their sums N and $\widetilde{N}$ are also mutually symmetric with respect to X. Here $N \setminus (X)$ and $\widetilde{N} \setminus (X)$ are open half-spaces bounded by the plane P (as follows from the fact that X is a limiting point for interior points of all the sets $t(M_i)$, $\tilde{t}(M_i)$, while the plane P is tangent to their boundaries). Therefore

$$P = [A_n \setminus (N \cup \widetilde{N})] \cup (X).$$

Now let $P' = f(P)$, $X' = f(X)$, $N' = f(N)$, $\widetilde{N}' = f(\widetilde{N})$. The sets N', $\widetilde{N}'$ are sums of f-images of the sets $t(M_i)$, $\tilde{t}(M_i)$. But by virtue of the conditions

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by (1) these images are translated sets $M_i' = f(M_i)$. They are centrally symmetric and pairwise have a single common point X'. Therefore they are pairwise symmetric with respect to X', and hence N' and $\widetilde{N}'$ are also symmetric with respect to X'.

The complement of the set $(N' \cup \widetilde{N}') \setminus (X')$ is the f-image of the plane P. Therefore $f(P)$ is also symmetric with respect to X'. But the point X was chosen arbitrarily on P, so the set $f(P)$ is symmetric with respect to each of its points. And since it is homeomorphic to P, it is itself an $(n-1)$-dimensional plane, and Theorem 1 is proved.

7. Let us turn to the proof of Theorems 2 and 3. Let M be a convex body, $\overline{\Gamma}$ a face of M, i.e. the intersection of M with some supporting plane P, and let Q be the plane spanned by $\overline{\Gamma}$. Replacing M by M_i, we may assume that M is centrally symmetric. We shall prove that $f(Q)$ is a plane.

If $\overline{\Gamma}$ is a point, the assertion is trivial. Therefore suppose that

$$k = \dim \overline{\Gamma} > 0,$$

and call the face Γ the interior of $\overline{\Gamma}$ relative to Q. Let $\tilde{\Gamma}$ and $\widetilde{P}$ be the face and supporting plane symmetric to Γ and P. Subject M to such a translation t_0 that $t_0(\widetilde{P}) = P$ and $t_0(\tilde{\Gamma})$ intersects Γ.

Taking a small k-dimensional ball $V \subset \tilde{\Gamma} \cap t_0(\Gamma)$, we may subject M to arbitrary small translations t along Q such that $t(M) \supset V$. In other words, in the group T_Q of translations parallel to Q, there is a neighborhood of zero U such that, for $t \in U$, $t(\Gamma) \supset V$. Under the mapping f we obtain the sets

$$ft_0(M) = t_0'(M')$$

and

$$ft(M) = t'(M').$$

Since f is a homeomorphism, $t_0'(M') \cap t'(M')$ are k-dimensional, just as $t_0(M) \cap t(M)$.

The homeomorphism f defines a homeomorphism φ of the group T onto T':

$$\varphi(t) = t',$$

where t' corresponds to t by virtue of relation (1), and moreover $\varphi(0) = 0$. Therefore the set $\varphi(U)$ is a neighborhood of zero of the group T' relative to $\varphi(T_Q)$. To the translations $t \in U$ for which $t(\Gamma) \supset V$, there correspond translations t' for which $ft(\Gamma)$ covers $f(V) \subset f(\overline{\Gamma})$. Such sufficiently small translations form a local group—a neighborhood of zero in some subgroup of the entire group T'.

It follows that, at least near zero, $\varphi(U)$ is such a local group. This means that $f(\Gamma)$ admits a locally free parallel displacement along $f(V)$. Consequently, $f(V)$ is a domain in a k-dimensional plane.

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But since we could choose the translation t_0 so that $t_0(\tilde{\Gamma})$ covered a neighborhood V of any given point in Γ, it follows that $f(\Gamma)$ is an open set in a k-dimensional plane. Translating M and $t_0(M)$ along the plane Q, we see that $f(Q)$ itself is a plane of the same dimension.

8. Under the assumptions of item 7, let us prove that if P is a plane parallel to a tangent plane to M, then $f(P)$ is also a plane.

Let P_0 touch M along the face Γ, and let Q_0 be the plane of this face. Let $P \parallel P_0$ and $x \in P$. Subject each set M_i to all such translations that its corresponding face falls on Q_0, and also to translations under which the face symmetric to it also falls on Q_0. Then the sums of such sets over all i form two sets $N, \widetilde{N}$, with

$$N \cap \widetilde{N} = Q,$$

and $N, \widetilde{N}$ mutually symmetric with respect to every point on Q and, in particular, the point X, while $N \setminus Q, \widetilde{N} \setminus Q$ are open half-spaces bounded by P. Now, applying the mapping f and using the fact that $f(Q)$ is also a k-dimensional plane, we see that $f(N)$ and $f(\widetilde{N})$ are mutually symmetric with respect to $f(X)$. Hence, just as in the proof of Theorem 1, we conclude that $f(P)$ is an $(n-1)$-dimensional plane.

9. **Proof of Theorem 2.** If a convex body is not a cylinder, then it has no fewer than $(n+1)$ tangent planes in general position. Therefore Theorem 3 follows from what was proved in item 8.

10. **Proof of Theorem 3.** Let the convex body M be the cylinder (5). It has faces parallel to each generator l_i. Therefore, by what was proved in § 7, the lines $L_i \parallel l_i$ are mapped into lines. The set N is a face of M, and therefore the plane Q spanned by it is mapped into a plane; moreover, since N is not a cylinder, by Theorem 2, f is affine on Q and, of course, on all such parallel planes. From these observations we easily arrive at Theorem 3.

11. Let us prove the first part of Theorem 4, when M_i is open. We shall write M instead of M_i. Fix a point $X \in A_n$, and let $X' \in f(X)$, while U' is a neighborhood of X'. Let H be the closed convex hull of M'. Since M' is centrally symmetric, so is H. Let $B, \widetilde{B}$ be its mutually symmetric extreme points. It is necessary that $B, \widetilde{B} \in \overline{M_1'}$.

Let $t', \tilde{t}'$ be translations: $B \to X'$, $\widetilde{B} \to X'$. Then, since the points $B, \widetilde{B}$ are extreme,
$t'(\overline{M'}) \cap \tilde{t}'(\overline{M'}) = (X')$. In view of this, there are such small translations $\tau, \tilde{\tau}$ that

$$X' \in \tau t'(M') \cap \tilde{\tau}\tilde{t}'(M) \subset U'. \tag{7}$$

To these translations $\tau t'$, $\tilde{\tau}\tilde{t}'$, by virtue of (1), there correspond such translations $t, \tilde{t}$ that

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$\tau t'(M') = ft(M)$ and similarly for $\tilde{\tau}\tilde{t}'$. Therefore from (7) it follows (since f is one-to-one)

$$f(X) \in f(t(M) \cap \tilde{t}(M)) \subset U'. \tag{8}$$

But M is open, and therefore $t(M) \cap \tilde{t}(M)$ is a neighborhood of X. Thus the continuity of f has been proved.

12. Let us prove the second part of Theorem 4, when M is closed with interior points and f^{-1} is bounded. Since M has interior points, the sets M_i together cover all of A_n. Therefore their f-images M_i' cover A_n'. If, moreover, no M_j' covered a given bounded domain, then f^{-1} would not be bounded. Consequently, for some j, M_j' has interior points. For brevity denote this M_j' by M' and M_j by M.

Take any point $X' \in A_{m'}$ and about it an open ball U so small that it is covered by some set $t'(M')$. Let r be the radius of U. Form the set V, the intersection of all $t'(M') \supset U$. It contains no balls equal to U (except U itself), since if U' is such a ball and τ is the translation from its center to the center of U, then $\tau(V) \supset U$ and, consequently, by the definition of the set V, it must be that $\tau(V) = V$. But then V is unbounded, contrary to the boundedness of M'.

Thus, V contains no balls equal to U. Hence it follows that outside the ball $2U$, concentric with U and of double radius, the set V contains no open balls U' of radii $r' \geq r - \varepsilon$ with some $\varepsilon > 0$.

Take a ball U_1 of radius $r_1 = \varepsilon/2$, concentric with U. Define from it the set V_1, as V is defined from U. Then, in view of the choice of r_1, it turns out that $V_1 \subset 2U$. Continuing this process, we obtain a sequence of sets V_i:

$$U_i \subset V_i \subset 2U_{i-1}, \qquad \bigcap V_i = (X').$$

Now observe that, since V_i is the intersection of all $t'(M') \supset U_i$, each $f^{-1}(V_i)$ is the intersection of all $f^{-1}t'(M') \supset f^{-1}(U_i)$. By relation (1), $f^{-1}t'(M) = t(M)$, and, by assumption, M is closed and bounded, i.e. compact. Therefore the sets $f^{-1}(V_i)$ are compact. But their intersection is the point $X = f^{-1}(X')$. Therefore, whatever neighborhood W of the point X is chosen, there is an i such that $f^{-1}(V_i) \subset W$. This proves the continuity of f^{-1}, and hence also of f.

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REFERENCES

1. A. D. Aleksandrov, *DAN*, **190**, No. 3 (1970).

Note: Figure translations are in progress. See original paper for figures.

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