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ON THE SOLVABILITY OF QUASILINEAR PARABOLIC SYSTEMS

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This note contains an application of the theorem on the solvability of a nonlinear equation in a Banach space from paper ([1]) to boundary-value problems for quasilinear parabolic (in the sense of Douglis–Nirenberg–Solonnikov ([2])) systems of differential equations. For simplicity in formulating the theorem we have restricted ourselves to the case when all right-hand sides of the system belong to the space $L_p(Q)$, and to the case of linear boundary conditions.

It turns out that, for the solvability of the boundary-value problems considered below for quasilinear parabolic systems, it is sufficient that the corresponding a priori estimate for the solutions of these problems exist. Thus, in this case there is no need, after obtaining the a priori estimate, to carry out an additional proof of the existence of a solution.

In the note we make essential use of the results of V. A. Solonnikov ([3]) on boundary-value problems for linear parabolic systems.

1. Definitions. Let Ω be a bounded domain in the space R^n with boundary S, $Q = \Omega \times [0, T]$, $0 < T < +\infty$, and $\Gamma = S \times [0, T]$. Let $u(x, t) = (u_1(x, t), \dots, u_m(x, t))$ be a real vector-function. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ be an integer nonnegative multi-index, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n$, $D^\alpha u_k = \partial^{|\alpha|} u_k(x, t) / \partial t^{\alpha_0} \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. Put $|\alpha|_b = 2b\alpha_0 + \alpha_1 + \dots + \alpha_n$, where b is some positive integer.

Consider a quasilinear system of equations of the form

$$\sum_{j=1}^{m} \sum_{|\gamma|_b = t_j} a_{ij}^{\gamma}(x, t, D^{\alpha^k} u_k)\, D^\gamma u_j + a_i(x, t, D^{\alpha^k} u_k) = f_i(x, t). \tag{1}$$

Here $i, k = 1, \dots, m$; $t_j = 2bt'_j$, where t'_j $(j = 1, \dots, m)$ are nonnegative integers and $\sum_{j=1}^{m} t_j = 2br$ (r is a positive integer); γ and α^k are integer nonnegative multi-indices with $|\alpha^k|_b \le t_k - 1$; a_{ij}^γ, a_i, and f_i are real functions. In addition, the functions a_{ij}^γ and a_i contain neither $D^{\alpha^k} u_k$ nor u_k if $t_k = 0$.

To the system (1) we assign boundary conditions of the form

$$\sum_{j=1}^{m} \sum_{|\gamma|_b \le \sigma_s + t_j} b_{sj}^{\gamma}(x,t)\, D^{\gamma} u_j \Big|_{\Gamma} = \Phi_s(x',t). \tag{2}$$

Here $s = 1, \dots, br$; σ_s are integers with $\max_s \sigma_s < 0$; γ is an integer nonnegative multi-index; b_{sj}^{γ} and Φ_s $(x' \in S)$ are real functions. In addition, $b_{sj}^{\gamma} \equiv 0$ if $\sigma_s + t_j < 0$.

We consider the system (1) under the boundary conditions (2) and the initial zero conditions

$$\partial^i u_j / \partial t^i \Big|_{t=0} = 0, \quad i = 0, \dots, t_j' - 1; \quad j = 1, \dots, m. \tag{3}$$

At the same time, if $t_j' = 0$, then no initial condition is imposed for u_j. With the system (1) with conditions (1) and (3) we associate the real spaces

$$H_1(Q) = \prod_{j=1}^{m} \dot{W}_p^{t_j, t_j'}(Q), \qquad H_2(Q) = \prod_{i=1}^{m} L_p(Q)$$

and

$$H_3(\Gamma) = \prod_{s=1}^{br} \dot{W}_p^{-\sigma_s - \frac{1}{p},\, \frac{1}{2b}\left(-\sigma_s - \frac{1}{p}\right)}(\Gamma),$$

with $p > n + 2b$ and with norms

$$\|u\|_{H_1(Q)} = \sum_{j=1}^{m} \|u_j\|_{p,Q}^{(t_j)}, \qquad \|f\|_{H_2(Q)} = \sum_{i=1}^{m} \|f_i\|_{p,Q} \quad \text{and} \quad \|\Phi\|_{H_3(\Gamma)} = \sum_{s=1}^{br} \|\Phi_s\|_{p,\Gamma}^{(-\sigma_s - 1/p)}.$$

The spaces $\dot{W}_p^{2bl,l}(Q)$ with integer $l \ge 0$ and $\dot{W}_p^{k,\frac{1}{2b}k}(\Gamma)$ with noninteger $k > 0$ are defined in the work of V. A. Solonnikov [3].

2. Solvability theorem. Problem (1)–(2)–(3) is considered under the following assumptions:

Condition I (smoothness condition). Let the real functions $a_{ij}^{\gamma}(x,t,D^{\alpha_k} u_k)$ and $a_i(x,t,D^{\alpha_k} u_k)$ be continuous and have continuous first derivatives with respect to the variables corresponding to $D^{\alpha_k} u_k$, for $(x,t) \in \overline{Q}$ and for arbitrary real values of the variables corresponding to $D^{\alpha_k} u_k$. Let the real coefficients $b_{sj}^{i\gamma}$ of the boundary operators belong to the class

$$C_{x,t}^{-\sigma_s-\frac{1}{p}+\varepsilon,\ \frac{1}{2b}\left(-\sigma_s-\frac{1}{p}\right)}(\Gamma)$$

with $\varepsilon > 0$. Let the boundary S belong to the class $C^{t_{\max}}$ ($t_{\max} = \max_j t_j$).

Condition II (parabolicity condition). Let the linear (with respect to v) system

$$\sum_{j=1}^{m} \sum_{|\gamma|_b = t_j} a_{ij}^{\gamma}(x, t, D^{\alpha_k} u_k) D^{\gamma} v_j = \psi_i(x, t) \qquad (i = 1, \dots, m) \tag{4}$$

for every fixed vector-function $u(x,t)$ from $H_1(Q)$ be parabolic in the sense of work [2] (with $s_i = 0$) (see also [4], Ch. VII, §§ 8, 9).

Condition III (complementarity condition). Let, for every fixed vector-function $u(x,t)$ from $H_1(Q)$, the boundary conditions (2) satisfy the complementarity condition in the sense of work [2] (see also [4], Ch. VII, §§ 8, 9) with respect to the linear (in v) system (4).

Condition IV (condition for the existence of an a priori estimate). Let, for every possible solution $u(x,t) \in H_1(Q)$ of problem (1), (2), (3) with $\|f\|_{H_2(Q)} + \|\Phi\|_{H_3(\Gamma)} \le K$, the inequality $\|u\|_{H_1(Q)} \le C(K)$ hold, where $C(K) < +\infty$ for $K < +\infty$.

Taking into account the definitions, we formulate the solvability theorem for problem (1), (2), (3).

Theorem. *Let Conditions I, II, III, IV be satisfied and let $p > n + 2b$. Then for any vector-functions $f(x,t) \in H_2(Q)$ and $\Phi(x',t) \in H_3(\Gamma)$ there exists a solution $u(x,t) \in H_1(Q)$ of problem (1), (2), (3).*

The proof of the theorem is based on the use of Theorem 2 from work [1] and the results of V. A. Solonnikov [3] on the solvability of boundary-value problems for linear parabolic systems. We note that, in the proof of the theorem, new norms, equivalent to the original ones, are introduced in the spaces $H_2(Q)$ and $H_3(\Gamma)$ in order to ensure uniform convexity of the Banach space $H_2(Q) \times H_3(\Gamma)$.

Remark 1. If, in addition, one assumes that for any fixed $u(x,t)$ and $w(x,t)$ from $H_1(Q)$ the system linear with respect to $v(x,t)$

$$\sum_{j=1}^{m} \sum_{|\gamma|_b = t_j} \int_0^1 a_{ij}^{\gamma}(x, t, D^{\alpha_k} u_k + \tau D^{\alpha_k} w_k)\, d\tau \cdot D^{\gamma} v_j = \psi_i(x,t) \qquad (i = 1, \dots, m)$$

is parabolic in the sense of Douglis–Nirenberg–Solonnikov and the boundary conditions (2) satisfy the complementing condition (see (2); (4), Ch. VII, §§ 8, 9) with respect to this system, which is linear in v, then the assertion on uniqueness in the space $H_1(Q)$ of the solution of problem (1), (2), (3) is valid.

Remark 2. In the case when the order of the derivatives $D^{\alpha^k} u_k$ entering into the coefficients $a_{ij}^{\gamma}(x, t, D^{\alpha^k} u_k)$ is such that $|\alpha^k|_b < t_k - 1$, the exponent p may be decreased to an exponent $p_1 > 1$ such that $|\alpha^k|_b < t_k - (n + 2b)/p_1$ for $k = 1, \dots, m$. In this case the functions $a_i(x, t, D^{\beta^k} u_k)$ (β^k are integral nonnegative multiindices with $|\beta^k|_b \le t_k - 1$) and their derivatives with respect to the variables corresponding to $D^{\beta^k} u_k$ must satisfy power-growth conditions with respect to $D^{\gamma^k} u_k$ with $|\gamma^k|_b \ge t_k - (n + 2b)/p_1$ ($|\gamma^k|_b \le t_k - 1$).

Example. Let Ω be a bounded interval $[a, b] \subset \mathbf{R}$. Consider in the rectangle $[a, b] \times [0, T]$, with arbitrary $T > 0$ ($T < +\infty$), the following boundary-value problem:

$$\frac{\partial u_i}{\partial t} - \sum_{k=1}^{m} a_{ik}(x, t, u) \frac{\partial^2 u_k}{\partial x^2} + a_i(t, u) \left(\frac{\partial u_i}{\partial x} \right)^2 + b_i(x, t, u, u_x) = f_i(x, t), \quad (5)$$

$$u_i(a, t) = u_i(b, t) = 0 \quad \text{for } t \in [0, T], \quad (6)$$

$$u_i(x, 0) = 0 \quad \text{for } x \in [a, b], \quad (7)$$

where $u = (u_1(x, t), \dots, u_m(x, t))$, $u_x = (\partial u_1(x, t)/\partial x, \dots, \partial u_m(x, t)/\partial x)$, $i = 1, \dots, m$.

Here $a_{ik}(x, t, u)$ ($i, k = 1, \dots, m;\ u = (u_1, \dots, u_m)$) are real continuous functions with continuous first derivatives with respect to the variables $u_1, \dots, u_m$ for $(x, t, u) \in [a, b] \times [0, T] \times \mathbf{R}^m$, and such that, for any $(x, t, u) \in [a, b] \times [0, T] \times \mathbf{R}^m$ and $\xi = (\xi_1, \dots, \xi_m) \in \mathbf{R}^m$, the inequality

$$\sum_{i,k=1}^{m} a_{ik}(x, t, u) \xi_k \xi_i \ge a_0(|u|) |\xi|^2 \quad \text{with} \quad a_0(|u|) \ge c_0 > 0,$$

holds, where c_0 is a constant, $a_0(\rho)$ is a continuous function for $\rho \ge 0$, $|v|^2 = \sum_{i=1}^m v_i^2$ for $v = (v_1, \dots, v_m) \in \mathbf{R}^m$; $a_i(t, u)$ are real continuous functions with continuous derivatives with respect to the variables $u_1, \dots, u_m$ for $(t, u) \in [0, T] \times \mathbf{R}^m$, with $a_i(t, 0) \equiv 0$ when $u_1 = \dots = u_m = 0$, and such that for any $(t, u) \in [0, T] \times \mathbf{R}^m$ and $\xi \in \mathbf{R}^m$ the inequality

$$\sum_{i,k=1}^{m} \frac{\partial a_i(t, u)}{\partial u_k} \xi_i^3 \xi_k \ge a(|u|) \sum_{i=1}^{m} \xi_i^4 \quad \text{with} \quad a(|u|) \ge 0,$$

holds, where $a(\rho)$ is a continuous function for $\rho \ge 0$; $b_i(x, t, u, p)$ ($p = (p_1, \dots, p_m)$) are real continuous functions with continuous derivatives with respect to the variables $u_1, \ldots, u_m$ and $p_1, \ldots, p_m$ for $(x,t,u,p) \in [a,b] \times [0,T] \times \mathbf{R}^m \times \mathbf{R}^m$, and such that for any $(x,t,u,p) \in [a,b] \times [0,T] \times \mathbf{R}^m \times \mathbf{R}^m$ the inequality

$$\sum_{i=1}^{m} b_i^2(x,t,u,p) \le a_0(|u|) \left[c\left(|p|^2 + |u|^2 + 1\right) + \left(\frac{4}{3} - \varepsilon\right) a(|u|) \sum_{i=1}^{m} p_i^4 \right],$$

holds, where the constant $c > 0$ and ε is an arbitrarily small positive number.

For the boundary-value problem (5), (6), (7), with $f_i(x,t) \in L_4(Q)$ $(i = 1, \ldots, m;\ Q = [a,b] \times [0,T])$, there exists an a priori estimate for $\|u\|_{\overset{\circ}{W}{}_4^{2,1}(Q)}$.

$$\left(\|u\|^4_{\overset{\circ}{W}{}_4^{2,1}(Q)} = \sum_{i=1}^{m} \|u_i\|^4_{\overset{\circ}{W}{}_4^{2,1}(Q)} \right).$$

Applying the theorem and Remark 1 to this theorem, we obtain the following assertion.

If the assumptions indicated here are satisfied, then the boundary-value problem (5), (6), (7), for any fixed vector function $f(x,t) = (f_1(x,t), \ldots, f_m(x,t))$ with $f_i(x,t) \in L_4(Q)$ $(i = 1, \ldots, m)$, has a unique solution $u(x,t)$ with $u_i(x,t) \in \overset{\circ}{W}{}_4^{2,1}(Q)$ $(i = 1, \ldots, m;\ Q = [a,b] \times [0,T])$.

Remark. In connection with this example we note that a broad class of quasilinear (and nonlinear) parabolic systems with one spatial variable was considered by S. N. Kruzhkov ([5]).

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Note: Figure translations are in progress. See original paper for figures.