

# **SURFACES OF NEGATIVE EXTERNAL CURVATURE IN A RIEMANNIAN SPACE WITH NONPOSITIVE RIEMANNIAN CURVATURE**

MATHEMATICS

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.38153>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 513.736.3

*MATHEMATICS*

**I. S. BRANDT**

## **SURFACES OF NEGATIVE EXTERNAL CURVATURE IN A RIEMANNIAN SPACE WITH NONPOSITIVE RIEMANNIAN CURVATURE**

*(Presented by Academician P. S. Aleksandrov, March 16, 1970)*

**1°.** In the paper <sup>(1)</sup> N. V. Efimov established the following theorem: Let there be, in three-dimensional Euclidean space, a complete (in the sense of the intrinsic metric) surface  $\Sigma$  with negative Gaussian curvature  $K$ . Let  $k = \sqrt{-K}$  and  $q = \sup_{\Sigma} \left| \text{grad} \frac{1}{k} \right|$ . Then  $q > 1/\sqrt{3}$ .

Later, using another method, N. V. Efimov proved that in fact  $q = \infty$  (see <sup>(2)</sup>).

The results set forth below are, in a certain sense, a generalization of these facts to the case of a Riemannian space.

Let there be, in a three-dimensional Riemannian space  $R$  of nonpositive Riemannian curvature, a complete (in the sense of the intrinsic metric) surface  $\Sigma$  of negative external curvature  $K_e$ .

**Theorem 1.** *Under the conditions stated above,*

$$\sup_{\Sigma} \left\{ \left| \text{grad} \frac{1}{k} \right| + \frac{\Lambda - \lambda}{2k^2} \right\} = q > \frac{1}{\sqrt{3}}.$$

Here  $k = \sqrt{-K_e}$ ;  $\Lambda$  and  $\lambda$  are the greatest and least Riemannian curvatures of the space  $R$  at the point of the surface where the curvature  $K_e$  is computed.

**Theorem 2.** *There exist a Riemannian space  $R$  and a surface  $\Sigma$  satisfying the conditions of Theorem 1, for which  $q = 4.5$ .*

**2°.** Let, in a three-dimensional Riemannian space  $R$  with coordinates  $z^1, z^2, z^3$ , a surface  $\Sigma$  be given by

$$z^1 = z^1(x, y), \quad z^2 = z^2(x, y), \quad z^3 = z^3(x, y).$$

We shall write the two fundamental forms of the surface in the form:

$$I = E dx^2 + 2F dx dy + G dy^2,$$

$$II = L dx^2 + 2M dx dy + N dy^2 = -(dz dn).$$

Here  $n = (n^1, n^2, n^3)$  denotes the vector of the unit normal to the surface. In what follows we shall also need the vectors  $\xi_1(\xi_1^1, \xi_1^2, \xi_1^3)$  and  $\xi_2(\xi_2^1, \xi_2^2, \xi_2^3)$ ;  $\xi_1^i = \partial z^i / \partial x$ ,  $\xi_2^i = \partial z^i / \partial y$ . Let us denote by  $\tau_1(\tau_1^1, \tau_1^2, \tau_1^3)$  and  $\tau_2(\tau_2^1, \tau_2^2, \tau_2^3)$  the normalized tangent vectors:

$$\tau_1 = \xi_1 / |\xi_1|, \quad \tau_2 = \xi_2 / |\xi_2|.$$

By  $\tau_1^*(\tau_1^{*1}, \tau_1^{*2}, \tau_1^{*3})$  we shall denote the vector  $\tau_1$  rotated in the tangent plane to the surface through the angle  $\pi/2$  in the direction from  $\tau_1$  to  $\tau_2$ . Analogously we introduce the vector  $\tau_2^*(\tau_2^{*1}, \tau_2^{*2}, \tau_2^{*3})$ . We shall also use below the reduced coefficients of the second quadratic form:

$$\lambda = L / \sqrt{EG - F^2}, \quad \mu = M / \sqrt{EG - F^2}, \quad \nu = N / \sqrt{EG - F^2}.$$

We define the external curvature of the surface as follows:

$$K_e = \lambda\nu - \mu^2.$$

We shall denote by  $K_i$  the intrinsic curvature of the surface, and by  $K$  the curvature of the space in the direction of the tangent plane to the surface. The components of the curvature tensor of the space will be denoted by  $R_{lkpj}$ .

3°. The fundamental equations of the theory of surfaces (see, for example, (3)) can be rewritten in the form

$$K_i = K + K_e,$$

$$\lambda'_y - \mu'_x = a_1\lambda + b_1\mu + c_1\nu + C_1\sqrt{E},$$

$$\nu'_x - \mu'_y = a_2\lambda + b_2\mu + c_2\nu - C_2\sqrt{G}.$$

Here

$$C_1 = R_{lkpj}\xi_1^l\xi_2^k\tau_1^pn^j/\sqrt{EG - F^2}; \quad C_2 = R_{lkpj}\xi_1^l\xi_2^k\tau_2^pn^j/\sqrt{EG - F^2};$$

$a_1, b_1, c_1, a_2, b_2, c_2$  are certain coefficients depending on the intrinsic metric of the surface. The coefficients  $b_1$  and  $b_2$  are computed by the formulas

$$b_1 = 2\Gamma_{12}^2, \quad b_2 = 2\Gamma_{12}^1,$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols of the surface  $\Sigma$ .

Suppose that asymptotic coordinates  $(u, v)$  are introduced on the surface. We give the first quadratic form of the surface the form

$$ds^2 = e^2 du^2 + 2eg \cos \omega du dv + g^2 dv^2.$$

From the fundamental equations, after calculations similar to those carried out in <sup>(1)</sup>, one obtains the equations

$$\partial \ln(ek)/\partial s_2 = \sin \omega (\partial Q/\partial s_1^* + C_1^*/2k), \quad (\text{I})$$

$$\partial \ln(gk)/\partial s_1 = -\sin \omega (\partial Q/\partial s_2^* - C_2^*/2k).$$

Here  $Q = \frac{1}{2} \ln k$ ,  $\partial/\partial s_i$  is the differentiation operator in the direction of the vector  $\tau_i$ ,  $\partial/\partial s_i^*$  is the differentiation operator in the direction of the vector  $\tau_i^*$ ,  $C_1^* = R_{ikpj} \xi_1^l \xi_2^k \tau_1^{*p} n^j / eg \sin \omega$ ,  $C_2^* = R_{ikpj} \xi_1^l \xi_2^k \tau_2^{*p} n^j / eg \sin \omega$ .

The formulas for the geodesic curvatures in the coordinates of the lines are also generalized:

$$\chi_1 = -\partial \omega / \partial s_1 + \sin \omega (\partial Q / \partial s_2 - C_2 / 2k), \quad (\text{II})$$

$$\chi_2 = \partial \omega / \partial s_2 - \sin \omega (\partial Q / \partial s_1 + C_1 / 2k),$$

where  $\chi_1$  and  $\chi_2$  are the geodesic curvatures of the lines of the first and second families. Equations (I) and (II) in Euclidean space were first obtained by N. V. Efimov and E. G. Poznyak in <sup>(4)</sup>.

The following lemma holds; we state it without proof.

**Lemma.**  $|C_i| \leq (\Lambda - \lambda)/2$ ,  $|C_i^*| \leq (\Lambda - \lambda)/2$ .

From this lemma and from equations (I) and (II) it follows that surfaces for which

$$\sup_{\Sigma} \left\{ \left| \text{grad} \frac{1}{k} \right| + \frac{\Lambda - \lambda}{2k^2} \right\} = q, \quad (*)$$

have properties analogous to the properties of surfaces with a  $q$ -metric in Euclidean space.

In particular, the estimate for the difference of the side lengths of a net quadrilateral, due to E. R. Rozendorn (see <sup>(5)</sup>), is generalized.

The proof of Theorem 1 then proceeds analogously to the proof of Theorem 5 of <sup>(1)</sup>.

4°. To prove Theorem 2, consider a Riemannian space with metric

$$ds^2 = \left( \frac{1+u^2}{2y^2} \right) dx^2 - \frac{2u}{y^2} dx dy + \left( \frac{1+9u^2}{2y^2} \right) dy^2 + du^2,$$

$$-\infty < x < +\infty, \quad -\infty < u < +\infty, \quad y > 0.$$

It can be shown that the surface  $\Sigma\{u=0\}$  satisfies the conditions of Theorem 1, and for it  $q=4.5$ .

5°. Theorem 1, proved above, is indeed a generalization of Theorem 5 of <sup>(1)</sup>, since in the case of a space of constant curvature condition (\*) coincides with the definition of a  $q$ -metric. However, passing to a Riemannian space weakens, in a certain sense, the theorem on  $q$ -metrics, since the extrinsic curvature in condition (\*) will no longer be determined by the intrinsic metric.

Several corollaries can be derived from Theorem 1. For example, the following theorem is a special case of Theorem 1: if in a Riemannian space  $R$  of nonpositive curvature there is a surface, complete in the sense of the intrinsic metric, with extrinsic curvature  $K_e = -1$ , then there is a point on the surface where  $\Lambda - \lambda > 1$ . From Theorem 1 and from the example given in 4° it follows that the method of proof of Theorem 2 of <sup>(2)</sup> uses the Euclidean character of the space to a substantially greater degree than the method of proof of Theorem 5 of <sup>(1)</sup>.

From the equations derived one can also draw several conclusions. Thus, for example, when  $q < \sqrt[3]{2}$ , the surface, as in the Euclidean case, has a regular net of asymptotic lines.

Moscow Institute  
of Electronic Machine Building

Received  
4 III 1970

## References

- <sup>1</sup> N. V. Efimov, Uspekhi Mat. Nauk, **21**, issue 5 (131), No. 3 (1966).
- <sup>2</sup> N. V. Efimov, Mat. sbornik, **76**, issue 4, 499 (1968).
- <sup>3</sup> P. K. Rashevskii, *Riemannian Geometry and Tensor Analysis*, "Nauka," 1967.
- <sup>4</sup> N. V. Efimov, E. G. Poznyak, Dokl. Akad. Nauk SSSR, **137**, No. 1, 25 (1961).
- <sup>5</sup> E. R. Rozendorn, Dokl. Akad. Nauk SSSR, **149**, No. 4 (1963).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*