THE HYSTERANT OPERATOR

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THE HYSTERANT OPERATOR

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In recent years, at the Voronezh seminar on differential equations, much attention has been devoted to the study of problems with complex nonlinearities. In particular, problems with hysteresis-type nonlinearities have been studied. In this connection the authors considered various phenomenological models of elastic-plastic bodies and tried to give these models an adequate mathematical description. As it turned out, a simple and convenient mathematical description is admitted by a certain Prager–Ishlinskii model. Its description required the introduction of a new operation, in a certain sense analogous to the concept of the multiplicative integral.

Below the hysterant operator is described, its basic properties are indicated, and it is explained how, with the aid of this operator, to compute the stress in an elastic-plastic one-dimensional body. The elastic-plastic body is thereby regarded as a certain finite or infinite set of "fibers," each of which corresponds to a known Prager–Ishlinskii model (see, for example, the works ([1–6])).

1. Elementary hysterant. Let three numbers l_-, l_+, and l_0 be given, with $l_- \le l_0 \le l_+$. The number l_0 is assumed finite; for l_- the value $-\infty$ is allowed, and for l_+ the value $+\infty$.

Let $x(t)$ be a continuous piecewise-linear function defined on $[0, \infty)$. For each fixed $t > 0$, the interval $[0, t]$ is then divided by certain points $t_0 = 0, t_1, \dots, t_n = t$ into intervals $[t_{i-1}, t_i]$ on which $x(t)$ is linear. Introduce the notation

$$\Delta_i x = x(t_i) - x(t_{i-1})$$

and define a finite sequence of numbers $l(t_i)$ $(i = 0, 1, \dots, n)$ by induction, setting $l(t_0) = l_0$ and

$$l(t_i) = \begin{cases} l_-, & \text{if } l(t_{i-1}) + \Delta_i x < l_-, \\ l(t_{i-1}) + \Delta_i x, & \text{if } l_- \le l(t_{i-1}) + \Delta_i x \le l_+, \\ l_+, & \text{if } l(t_{i-1}) + \Delta_i x > l_+. \end{cases}$$

We shall call the operator $\Gamma[x]$, which assigns to the function $x(t)$ the function

$$\Gamma[x](t) = l(t_n). \tag{1}$$

the **elementary hysterant**.

The elementary hysterant has so far been defined on piecewise-linear functions. Obviously, the values of the hysterant do not depend on how $[0, t]$ is divided into intervals of linearity of the function $x(t)$. From the piecewise linearity of the function $x(t)$ it follows immediately that the function $\Gamma[x](t)$ is also piecewise linear.

Let now $x(t)$ $(0 \le t < \infty)$ be an arbitrary continuous function. Construct a sequence $x_k(t)$ of piecewise-linear functions converging to $x(t)$ uniformly on every finite interval.

Theorem 1. The sequence $\Gamma[x_k](t)$ converges uniformly on every finite interval.

This theorem makes it possible to define the elementary hysteron on every continuous function $x(t)$ by the equality

$$\Gamma[x](t) = \lim_{k\to\infty} \Gamma[x_k](t). \tag{2}$$

By Theorem 1, the limit (2) does not depend on the choice of the sequence $x_k(t)$ of piecewise-linear functions converging uniformly to x on every finite interval.

It follows from Theorem 1 that the operator (2) transforms every continuous function into a continuous one.

The value of the hysteron at the point t^* is very easily computed if the function $x(t)$ is piecewise monotone on the interval $[0, t^*]$. In this case it suffices to divide $[0, t^*]$ into parts by the points $t_1, \dots, t_{n-1}$ of minima and maxima, and to note that $\Gamma[x](t^*)$ coincides with $\Gamma[y](t^*)$, where $y(t)$ is a piecewise-linear function coinciding with $x(t)$ at the points $t_0 = 0, t_1, \dots, t_{n-1}, t_n = t^*$ and linear on each interval $[t_{i-1}, t_i]$.

We have defined the operator of an elementary hysteron on the space C of functions continuous on the half-line $[0, \infty)$. Clearly, $\Gamma[x](t) = \Gamma[y](t)$ for $0 \le t \le t_1$, if $x(t) = y(t)$ for $0 \le t \le t_1$. Therefore the hysteron may also be considered on functions specified only on some finite interval $[0, \tau]$.

The operator of the elementary hysteron depends, by definition, on the triple of numbers $\hat{l} = \{l_-, l_+, l_0\}$. Therefore we shall often include this triple of numbers in the notation of the hysteron and write either $\Gamma[x, \hat{l}](t)$, or $\Gamma[x, l_-, l_+, l_0](t)$.

2. Simplest properties. We shall use the notation

$$\|x\|_\tau = \max_{0\le t\le\tau} |x(t)|.$$

Theorem 2. The hysteron satisfies the following Lipschitz conditions:

$$|\Gamma[x, l_-, l_+, l_0](t) - \Gamma[x, l_-^*, l_+, l_0](t)| \leq |l_- - l_-^*|, \tag{3}$$

$$|\Gamma[x, l_-, l_+, l_0](t) - \Gamma[x, l_-, l_+^*, l_0](t)| \leq |l_+ - l_+^*|, \tag{4}$$

$$|\Gamma[x, l_-, l_+, l_0](t) - \Gamma[x, l_-, l_+, l_0^*](t)| \leq |l_0 - l_0^*|, \tag{5}$$

and, finally,

$$\|\Gamma[x, \hat{l}](t) - \Gamma[y, \hat{l}](t)\|_\tau \leq 2\|x - y\|_\tau. \tag{6}$$

The constant 2 in condition (6) and the constants 1 in conditions (3)–(5) cannot be improved.

The function $\Gamma[x, l_-, l_+, l_0](t)$, obviously, is nondecreasing in the variables l_-, l_+, and l_0; for any l_0 and l_0^* the difference $\Gamma[x, l_-, l_+, l_0](t) - \Gamma[x, l_-, l_+, l_0^*](t)$ does not increase as the variable t increases.

Introduce the notation $x_s(t) = x(t + s)$ $(t, s \geq 0)$.

Theorem 3. The identity holds

$$\Gamma[x_s, l_-, l_+, \Gamma[x, l_-, l_+, l_0](s)](t) = \Gamma[x, l_-, l_+, l_0](t + s). \tag{7}$$

From identity (7) and Theorem 2 it follows that

Theorem 4. Let $x, y \in C$ and $x(t) - y(t) \to 0$ as $t \to \infty$. Then there exists a finite limit

$$c = \lim_{t \to \infty} \{\Gamma[x, l_-, l_+, l_0](t) - \Gamma[y, l_-, l_+, l_0^*](t)\}. \tag{8}$$

3. Hysteron system. Consider a set M on which functions $l_-(a)$ and $l_+(a)$ are defined, with $l_-(a) \leq l_+(a)$. The functions $l_-(a)$ and $l_+(a)$ may take infinite values of the corresponding sign.

A finite function $l_0(\alpha)$ satisfying the inequalities

$$l_-(\alpha) \leq l_0(\alpha) \leq l_+(\alpha),$$

will be called a state of the system M. We shall assume that every function $x(t)$, continuous for $0 \leq t < \infty$, determines a change of the state of the system M according to the following law: for each fixed t, the state $l(\alpha, t)$ is determined by the formula

$$l(\alpha, t) = \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t). \tag{9}$$

In this case the set M, with the given functions $l_-(\alpha)$ and $l_+(\alpha)$, will be called a hysterant system.

Let some measure μ be given on M. All functions considered below are assumed measurable with respect to this measure. Further, let a function $F(\alpha, l)$ ($\alpha \in M$; $l_-(\alpha) \leq l \leq l_+(\alpha)$) be given, which is superposition-measurable in the sense that it transforms every measurable state $l(\alpha)$ into a function $F[\alpha, l(\alpha)]$ measurable on M.

The characteristic of the state $l(\alpha)$ of a hysterant system, or simply the characteristic of the system, will mean the functional

$$\Phi[l] = \int_M F[\alpha, l(\alpha)]\, d\mu(\alpha). \tag{10}$$

If the state of the system changes according to law (9), then the characteristic of the system will be a function of the variable t ($0 \leq t < \infty$). This function is determined by the equality

$$\Phi[x; l_0(\alpha)](t) = \int_M F\{a, \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\}\, d\mu(\alpha). \tag{11}$$

4. Model of a plastic body. As we have already said above, the concepts of a hysterant, a hysterant system, its state, and the characteristic of its state arise naturally in the mathematical description of models of elastic-plastic bodies.

We shall consider a one-dimensional elastic-plastic body as a set M of infinitesimal elastic "fibers" α, the length of each of which may vary within the limits from $l_-(\alpha)$ to $l_+(\alpha)$. The state of such a body is determined by the function $l(\alpha)$.

Let the elastic-plastic body be stretched or compressed, and let the function $x(t)$ be the variable length of the body. If each individual "fiber" corresponds to the Prager–Ishlinskii model, then formula (9) determines the lengths of the individual "fibers" at each instant of time t. In the model under consideration, the values of the function $x(t)$ are not restricted; if, for example, $x(t)$ takes values greater than the maximum length of the infinitesimal fibers, then this corresponds to the body "flowing," i.e., individual fibers being displaced relative to one another.

We emphasize that, in the scheme considered by us, the variable length $x(t)$ of the body under consideration may be an arbitrary continuous (and not necessarily piecewise monotone) function; this makes it possible to study more simply and more completely the dependence of changes in the states of the body on functions $x(t)$.

Formula (9) immediately explains why the state of an elastic-plastic body is determined not only by how much it is stretched or compressed, but also by the entire process $x(t)$ of deformation.

Formulas (10) and (11) make it possible to calculate the stress in an elastic-plastic body if the stress in each infinitesimal fiber α is equal to $F[\alpha, l(\alpha)]$, when it has length $l(\alpha)$. We shall dwell in more detail on the (apparently most important) special case.

Suppose that the length of each fiber in the unstressed state is equal to $l^*(\alpha)$ (if $l_-(\alpha)$ and $l_+(\alpha)$ are finite, then one may, for example, assume that

$l^*(\alpha) = \frac{1}{2}[l_-(\alpha) + l_+(\alpha)]$. Suppose, further, that in each infinitesimal fiber the stress varies according to Hooke' s law. Then it may be assumed that the infinitesimal moduli of elasticity are included in the measure μ and, consequently, the stress in an elastoplastic body in the state $l(\alpha)$ may be determined by the formula

$$\Phi[l] = \int_{\dot{M}} [l(\alpha) - l^*(\alpha)]\, d\mu(\alpha). \tag{12}$$

Formula (12) is a special case of formula (10). Formula (11), in the situation where Hooke' s law is applicable, takes the form

$$\Phi[x; l_0(\alpha)](t) = \int_M \{T[x, \hat{l}(\alpha)](t) - l^*(\alpha)\}\, d\mu(\alpha). \tag{13}$$

The function (13) determines the total stress in an elastoplastic body under any law of change of the length $x(t)$; in particular, the well-known hysteresis loops arise here. Analysis of the hysterant operator and of the state characteristics of a hysterant system explains such phenomena as residual deformation, the role of the initial state of the hysterant system, etc.

It should be noted that in the Prager-Ishlinskii model it is always assumed that the function $x(t)$, which determines the extension of an elastoplastic element, is piecewise monotone. This restriction, of course, is neither essential nor restrictive in the study of any fixed individual process. However, if one attempts to apply modern methods (integral equations, functional analysis, etc.) to the study of processes in systems with elastoplastic elements, it is convenient to be able to consider arbitrary continuous $x(t)$ determining extensions. The hysterant operator provides the possibility of such consideration.

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Note: Figure translations are in progress. See original paper for figures.