FROM THE EXISTENCE OF A NONMEASURABLE SET OF TYPE A_2 THERE FOLLOWS THE EXISTENCE OF AN UNCOUNTABLE SET CONTAINING NO PERFECT SUBSET OF TYPE CA

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Abstract

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FROM THE EXISTENCE OF A NONMEASURABLE SET OF TYPE A_2 THERE FOLLOWS THE EXISTENCE OF AN UNCOUNTABLE SET CONTAINING NO PERFECT SUBSET OF TYPE CA

(Presented by Academician P. S. Novikov, April 27, 1970)

The following problems of descriptive set theory are known: is assertion (I) true: "there exists an uncountable set of type CA containing no perfect subset" ;

is assertion (II) true: "there exists a set of type B_2 that is not Lebesgue measurable."

The formulation of these problems was refined in terms of axiomatic set theory. Namely, the question was posed as to the derivability of assertions (I) and (II), or of their negations, in one or another axiomatic theory—for example, in Zermelo-Fraenkel theory (denoted by ZF) or in Gödel-Bernays theory (denoted by GB or Σ). We note that, by virtue of general metatheorems of axiomatic set theory (see (^{1}), p. 149), derivability in ZF of assertions (I), (II), or of their negations is equivalent to the derivability of these same propositions in GB.

P. S. Novikov (^{2}) showed that the negations of propositions (I) and (II) cannot be derived in GB. In an abstract (^{3}), Solovay announced, without giving a proof, the existence of a model of ZF in which, in particular, assertions (I) and (II) are false. The author of the present note also constructed a model of ZF in which, in particular, assertion (I) is false, and reported this result at the All-Union Symposium on Mathematical Logic (Alma-Ata, June 2-7, 1969; see theorem 3a in the abstracts of that symposium (^{4})). At the symposium the following theorem was also reported (not included, however, in the abstracts (^{4})):

Theorem 1. *There exists a model of the theory ZF in which (II) is false and (I) is true.*

After a number of very diverse propositions that proved to be undecidable by the means of ordinary set theory (for example, propositions (I) and (II), the proposition that (II) follows from (I), etc.), it seems somewhat surprising that one classical question of descriptive set theory can be solved within the framework of the usual naive set theory without any additional assumptions.

Theorem 2. *It is derivable in ZF that from the existence of a nonmeasurable set of type A_2 there follows the existence of an uncountable set of type CA containing no perfect subset.*

Using the following criterion (see ([4,6])):
$ZF \vdash$ "there exists $u(L^{+}(u)$–uncountable $\equiv (I))$," Theorem 2 is easy to derive from the following more general assertion.

Theorem 2′.

$$ZF \vdash \neg u \neg \varphi(\varphi \in \Sigma_2^1(u) f\{x \mid \mathrm{Icm}\, \varphi(x)\}\text{–nonmeasurable}) \equiv R(u)\text{–not of measure } 0).$$

In the formulation of Theorem 2′ the following abbreviations are used:
$\varphi \in \Sigma_2^1(u) \cup \Pi_2^1(u)$ means that the formula φ contains only two quantifiers over real numbers (one existential quantifier and one quantifier ···

generality) and any number of quantifiers over natural numbers, prefixed to a predicate decidable relative to the real number u (for the exact definition see ([5])); $L^{+}(u)$ is a term which, from the real number u, yields the set of all real numbers constructive (in the sense of Gödel) relative to u. $\mathrm{Ist}\, \varphi$ is the formula ZF, having the following meaning: if the formula φ contains only quantifiers over sets of real or natural numbers, then $\mathrm{Ist}\, \varphi$ means that the formula φ is true. It is easy to see that for every φ, $\varphi \in \Sigma_2^1(u) \cup \Pi_2^1(u)$, the set $\{x \mid \mathrm{Ist}\, \varphi(x)\}$ is of type A_2 or CA_2. Conversely, every set of real numbers of type A_2 or CA_2 can be described in this sense by a formula φ, $\varphi \in \Sigma_2^1(u) \cup \Pi_2^1(u)$. $\overline{R}(u)$ is a term which, from the number u, yields the set of real numbers non-random relative to u.

When the problems formulated at the beginning of the present note were posed, the assertions themselves (I) and (II) were understood by some authors (and, first of all, by N. N. Luzin: see ([7]), pp. 553-556, 565) differently from assertions of pure existence. What was meant was "effective existence," i.e., whether one can effectively specify this or that object, even if one allows that such an object exists.

Theorem 3. *There is no term T with free variables $x_1 \ldots x_n$, for which in $ZF + (I)$ the assertion is provable: "$\exists a_1 \ldots \exists a_n$ ($a_1 \ldots a_n$ are ordinals and $T(a_1 \ldots a_n)$ is an A-set whose complement is uncountable and contains no perfect kernel)."*

We give a sketch of the proof of Theorem 3. First of all, from a term T_1 of type Π_1^1 one can construct a term T which specifies a sieve (see ([7]), p. 162) not precisely sifting the set $\{x \mid T_1(x)\}$. Suppose now the contrary: let the term $T(a_1 \ldots a_n)$, where $a_1 \ldots a_n$ are arbitrary ordinals, specify an uncountable CA-set containing no perfect kernel; more precisely

$$\underset{ZF}{\vdash}\ \exists a_1 \dots \exists a_n\ (a_1 \dots a_n \text{ are ordinals } \& T(a_1 \dots a_n) \text{ is}$$

an uncountable CA-set without a perfect kernel). In particular, it is derivable in ZF that the set $T(a_1 \dots a_n)$ is the union of an uncountable number of nonintersecting constituents, each of which is countable. P. S. Novikov indicated an effective method for choosing a point from a CA-set specified by a sieve (see (^{7}), p. 617). With the aid of this method, from the term $T(a_1 \dots a_n)$ one can construct another term $T'(a_1 \dots a_n)$ possessing the following property: for every ordinal α such that the α-th constituent of the sieve $T(a_1 \dots a_n)$ is nonempty, the term $T'(a_1 \dots a_n \alpha)$ is a real number from this constituent. The term $T'(a_1 \dots a_n \alpha)$ therefore specifies an uncountable well-ordered sequence of distinct real numbers, and this circumstance is derivable in ZF, i.e., is true in every model of ZF. Consider the model of ZF described in the book (^{1}), p. 267, taking as the α occurring there the first uncountable ordinal ω_1^M of the minimal model M.

We shall denote this model by N_h. It can be shown that in N_h proposition (I) is valid. The term T', relativized to N_h, gives an uncountable sequence of real numbers in N_h, definable in N_h by this term $(T')_N$. On the other hand, using the notion of permutation introduced by A. Lévy in (^{8}) in connection with another model, one can show that every sequence of real numbers definable in N_h is countable (in N_h). This gives a contradiction.

Corollary. *There is no term T without free variables for which in $ZF + (I)$ the assertion is provable: "T is an A-set whose complement is uncountable and contains no perfect kernel."*

Theorem 4. *There is no term T with free variables $x_1 \dots x_n$, for which in $ZF + (I) + (II)$ the assertion is provable: "$\exists a_1 \dots \exists a_n$ ($a_1 \dots a_n$ are ordinals and $T(a_1 \dots a_n)$ is a nonmeasurable set)."*

Corollary. There is no term T, without free variables, for which the assertion "T is a nonmeasurable set" is proved in $ZF + (I) + (II)$.

I wish to thank my supervisor V. A. Uspenskii, who set before me a number of interesting problems (answers to some of them are given here) and constantly helped me in my work, and A. G. Dragalin for repeated discussions of these theorems and for advice on improving their proofs.

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