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Abstract

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MATHEMATICS

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SOME LIMIT THEOREMS FOR LARGE DEVIATIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES

(Presented by Academician Yu. V. Linnik on 8 X 1969)

1. Consider a sequence of independent random variables $X_1, X_2, \ldots$ with finite variances $\sigma_1^2, \sigma_2^2, \ldots$, not all of which are zero. Without loss of generality we shall assume that the mathematical expectations are equal to zero: $EX_j = 0$ $(j = 1, 2, \ldots)$.

Introduce the following notation

$$B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad Z_n = B_n^{-1} \sum_{j=1}^{n} X_j, \qquad L_n = B_n^{-3} \sum_{j=1}^{n} EX_j^3,$$

$$F_n(x) = \mathbf{P}\{Z_n < x\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{t^2}{2}\right\} dt.$$

In the present note a number of integral and local theorems for large deviations are given for classes of functions introduced by Yu. V. Linnik (^{1}).

2. Consider nondecreasing functions $h(x)$, defined for $x > 1$ and belonging to one of the following three classes*.

Class I. $h(x)$–functions with continuous first derivatives satisfying the conditions

$$(\ln x)^{2+\zeta_0} \le h(x) < x^{1/2},$$

where ζ_0 is a positive constant which may be arbitrarily small.

Next, put

$$h(x) = \exp\{H(\ln x)\}$$

and impose on $H(z)$ the following conditions: $H(z)$ is a monotone differentiable function; $H'(z) \leq 1$; $H'(z) \to 0$ as $z \to \infty$; $H'(z)\exp\{H(z)\} > c_1 z^{1+\zeta_1}$, where c_1 and ζ_1 are positive constants.

Class II. $h(x)$—functions satisfying the conditions:

$$\rho_0(x)\ln x \leq h(x) \leq (\ln x)^2,$$

$$h(x) = M(x)\ln x = N(\ln x)\ln x,$$

$N'(z)$ is monotone; $N'(z) \to 0$ as $z \to \infty$; $\rho_0(x)$ is a function increasing to ∞ arbitrarily slowly.

Class III. $h(x)$ functions satisfying the conditions

$$3\ln x \leq h(x) \leq K\ln x,$$

where $K \geq 3$ is a constant.

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* In ([1]), the continuity of $h'(x)$ from class I and the monotonicity of $N'(z)$ from class II are not assumed to hold.

Let us introduce the condition

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E\exp\{h(|X_j|)\} < \infty. \tag{1}$$

Define the function $\Lambda(n)$ for each of the three classes considered above by means of the equations

$$h(\sqrt{n}\Lambda(n)) = (\Lambda(n))^2, \tag{2}$$

$$\sqrt{h(n)} = \sqrt{M(n)}\ln n = \Lambda(n), \tag{3}$$

$$\Lambda(n) = \sqrt{\ln n} \tag{4}$$

respectively.

3. Let a function $h(x)$ of class I be given.

Theorem 1. *Suppose that conditions (1), (2), and the condition*

$$\lim_{n\to\infty} \frac{B_n^2}{n} > 0 \tag{5}$$

are satisfied. Then

$$\frac{1-F_n(x)}{1-\Phi(x)} = \exp\left\{\frac{x^3}{6}L_n\right\}\left[1+O\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{6}$$

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left\{-\frac{x^3}{6}L_n\right\}\left[1+O\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{7}$$

as $n \to \infty$ in the region $0 \le x \le \Lambda(n)/\rho(n)$, where $\rho(n)$ is an arbitrary function satisfying the condition

$$\lim_{n\to\infty} \rho(n) = \infty. \tag{8}$$

Corollary. *Suppose that the conditions of Theorem 1 are satisfied. Then*

$$\lim_{n\to\infty} \frac{1-F_n(x)}{1-\Phi(x)} = 1, \qquad \lim_{n\to\infty} \frac{F_n(-x)}{\Phi(-x)} = 1$$

in the region $0 \le x \le \Lambda(n)/\rho(n)$, whatever the function $\rho(n)$ satisfying condition (8).

Theorem 2. Suppose

$$\lim_{n\to\infty} \frac{B_n^2}{n} < \infty \tag{9}$$

and $\Lambda(n)$ is determined from (2). Suppose, furthermore, that there exist positive constants b_1 and b_2 and a function $\rho(n)$ satisfying condition (8) such that

$$1-F_n(x) \le b_1 e^{-b_2 x^2}, \qquad F_n(-x) \le b_1 e^{-b_2 x^2} \tag{10}$$

for $0 \le x \le \Lambda(n)\rho(n)$ and all sufficiently large n. Then

$$E\exp\{h(|X_j|)\} < \infty \tag{11}$$

for all j.

Analogous theorems are valid for functions $h(x)$ of class II.

Let a function $h(x)$ of class III be given.

Theorem 3. *Suppose that conditions (1), (4), and (5) are satisfied. Then*

$$\frac{1-F_n(x)}{1-\Phi(x)} \to 1, \qquad \frac{F_n(-x)}{\Phi(-x)} \to 1$$

as $n \to \infty$ in the region $0 \le x \le \Lambda(n)/\rho(n)$, where $\rho(n)$ is an arbitrary function satisfying condition (8).

A statement analogous to Theorem 2 is also valid.

4. We give the statements of the corresponding local theorems. Introduce the notation:

$$\nu_j(t) = Ee^{itX_j}, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}.$$

Denote by $p_n(x)$ the derivative of the distribution function $F_n(x)$, if $F_n(x)$ is absolutely continuous.

Let a function $h(x)$ of class I be given.

Theorem 4. *Suppose that conditions (1), (2), and (5) are satisfied. Suppose, furthermore, that to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that*

$$\int_{|t|>\varepsilon} \prod_{j=1}^{n} |\nu_j(t)|\, dt = O\left(e^{-\delta(\Lambda(n))^2}\right) \qquad (n \to \infty).$$

Then, for all sufficiently large n, there exists everywhere a continuous density $p_n(x)$ of the distribution of the random variable z_n; moreover,

$$\frac{p_n(x)}{\varphi(x)} = \exp\left\{\frac{x^3}{6} L_n\right\} \left[1 + O\left(\frac{|x|+1}{\sqrt{n}}\right)\right] \tag{12}$$

as $n \to \infty$ in the domain $|x| \le \Lambda(n)/\rho(n)$, where $\rho(n)$ is an arbitrary function satisfying condition (8).

Corollary. *Suppose the conditions of Theorem 4 are satisfied. Then*

$$\lim_{n\to\infty} \frac{p_n(x)}{\varphi(x)} = 1$$

in the domain $|x| \le \Lambda(n)/\rho(n)$, whatever the function $\rho(n)$ satisfying condition (8).

Theorem 5. *Suppose condition (9) is satisfied, $\Lambda(n)$ is defined by (2), and the random variable z_n, for some $n = n_0$, has a continuous distribution with density*

$p_n(x)$. *Suppose, furthermore, that there exist positive constants* b_0 *and* b_1 *and a function* $\rho(n)$ *satisfying condition (8), such that*

$$p_n(x) \le b_0 e^{-b_1 x^2} \tag{13}$$

for $|x| \le \Lambda(n)\rho(n)$ *and all sufficiently large* n. *Then condition (11) is satisfied for all* i.

For classes II and III, the corresponding local limit theorems for densities, analogous to the integral theorems, are valid.

5. We note that (10) and (13) hold, respectively, in the domains $0 \le x \le \Lambda(n)\rho(n)$ and $|x| \le \Lambda(n)\rho(n)$ for a sufficiently slowly increasing function $\rho(n)$, if, respectively, relations (6), (7), and (12) hold in these domains and if the expression $|\sqrt{n}\, L_n|$ is bounded.

6. The results of the present note are a continuation of the investigations of works ($^{1-3}$).

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CITED LITERATURE

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[2] V. V. Petrov, *Vestn. Leningrad. Univ.*, No. 19, 49 (1963); No. 1, 58 (1964).
[3] V. Wolf, *DAN*, 178, No. 1, 21 (1968).

Note: Figure translations are in progress. See original paper for figures.

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