ON FREDHOLM THEORY FOR NONLINEAR OPERATOR EQUATIONS

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Abstract

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The Fredholm theory and the generalization of this theory to linear operator equations in Banach spaces are widely known. As simple examples show, this theory does not carry over to arbitrary nonlinear equations with completely continuous operators.

In the present note we have succeeded in showing that some of the Fredholm theorems are partially carried over to nonlinear operator equations with completely continuous operators having an asymptotic derivative. The proof is carried out with the aid of the Schauder–Tikhonov principle (^{1}). The results obtained are applied to nonlinear integral equations and boundary-value problems for nonlinear ordinary differential equations.

We note that equations with asymptotically differentiable operators were previously studied in other situations by M. A. Krasnosel’ skii (^{2}).

1. Let X be a Banach space; X^* its conjugate space; $A : X \to X$ a linear completely continuous operator; A^* its adjoint. Let I be the identity operator in X; (φ, y) the value of the linear functional $\varphi \in X^*$ at the element $y \in X$.

We introduce the usual notation:

$$\ker(I-A) = \{x:\ x = Ax,\ x \in X\}; \qquad \ker(I-A)^* = \{z:\ z = A^*z,\ z \in X^*\};$$

$$\{\ker(I-A)\}^{\perp} = \{z:\ z \in X^*,\ (z,x) = 0,\ \forall x \in \ker(I-A)\};$$

$$\{\ker(I-A)^*\}^{\perp} = \{y:\ y \in X,\ (\varphi,y) = 0,\ \forall \varphi \in \ker(I-A)^*\}.$$

Recall that a linear or nonlinear operator $T : X \to X$ is called completely continuous if it is continuous and maps bounded sets into compact ones.

A linear bounded operator $B : X \to X$ is called an asymptotic derivative of the operator T if

$$\lim_{\|x\| \to \infty} \|T(x) - Bx\| / \|x\| = 0.$$

Consider the system of nonlinear algebraic equations

$$\sum_{k=1}^{\mu} a_{ik}\xi_k + \varphi_i(\xi_1, \dots, \xi_\mu) = \eta_i, \qquad i = 1, \dots, \mu. \tag{1}$$

Here a_{ik} are real numbers, and $\varphi_i(\xi_1, \dots, \xi_\mu)$ are real functions. Consider the auxiliary adjoint linear homogeneous system

$$\sum_{i=1}^{\mu} a_{ik} t_i = 0, \qquad k = 1, \dots, \mu. \tag{2}$$

Lemma 1. *Let the continuous functions φ_i $(i = 1, \dots, \mu)$ of the real variables $\xi_1, \dots, \xi_\mu$ possess the properties:*

(a) there exists a number $p > 1$ such that

$$\lim_{\sum_{i=1}^{\mu} |\xi_i|^p \to \infty} \frac{\sum_{i=1}^{\mu} |\varphi_i(\xi_1, \dots, \xi_\mu)|^p}{\sum_{i=1}^{\mu} |\xi_i|^p} = 0;$$

(b) for any vector $\{t_1^0, \dots, t_\mu^0\}$ satisfying system (2), the equality

$$\sum_{i=1}^{\mu} \varphi_i(\xi_1, \dots, \xi_\mu) t_i^0 = 0$$

holds, whatever the vector $\{\xi_1, \dots, \xi_\mu\}$ may be.

If the conditions listed above are fulfilled, then for the solvability of system (1) for a given vector $\{\eta_1, \dots, \eta_\mu\}$ it is necessary and sufficient that this vector be orthogonal to all solutions of system (2).

This lemma generalizes the well-known result on the solvability of systems of linear algebraic equations. With the aid of this lemma and the Schauder–Tikhonov principle, one proves

Theorem 1. Let X be a Banach space, $T : X \to X$ an operator defined on all of X and completely continuous, having asymptotic derivative B. Let the operator $\Omega(x) = T(x) - Bx$ have the property

$$\Omega(x) \in \{\ker(I - B)^*\}^\perp, \quad \forall x \in X.$$

In order that, for a given $y \in X$, the equation $x = T(x) + y$ be solvable, it is necessary and sufficient that $y \in \{\ker(I - B)^*\}^\perp$.

A consequence of it is

Theorem 2. Let X be a Banach space, $T : X \to X$ an operator defined on all of X and completely continuous, having asymptotic derivative B. Let the equation $x = Bx$ have the unique solution $x = 0$ in the space X. Then the equation $x = T(x) + y$ is solvable in X for every $y \in X$.

Remark 1. If T is a linear operator, then the condition $\Omega(x) \in \{\ker(I-B)^*\}^\perp$, $\forall x \in X$, is always fulfilled. In this case, from Theorem 1 we obtain (for the case of integral operators) the third Fredholm theorem. Theorem 2 is a generalization of the first Fredholm theorem. There are examples of integral operators showing that, in the nonlinear case, the solution of the equation $x = T(x) + y$ under the conditions of Theorem 2 need not be unique.

Let us now consider the question of solvability of a nonlinear equation which is, in a certain sense, adjoint to the equation $x = T(x) + y$.

Theorem 3. Let X be a Banach space, $T : X \to X$ an operator defined on all of X and completely continuous, having asymptotic derivative B. Let $\widetilde{T} : X^* \to X^*$ be an operator defined on all of X^* and completely continuous, having as its asymptotic derivative the operator B^* adjoint to the operator B. Let the operator $\widetilde{\Omega}(v) = \widetilde{T}(v) - B^*v$ have the property $\widetilde{\Omega}(v) \in \{\ker(I - B)\}^\perp$, $\forall v \in X^*$.

In order that, for a given $\tilde{y} \in X^*$, the equation $v = \widetilde{T}(v) + \tilde{y}$ be solvable, it is necessary and sufficient that $\tilde{y} \in \{\ker(I - B)\}^\perp$.

Theorem 4. Let X be a Banach space, $T : X \to X$ an operator defined on all of X and completely continuous, having as its asymptotic derivative the operator B. Let $\widetilde{T} : X^* \to X^*$ be an operator defined on all of X^* and completely continuous, having as its asymptotic derivative the operator B^*, adjoint to the operator B. Let the equation $x = Bx$ have the unique solution $x = 0$ in the space X. Then the equation $v = \widetilde{T}(v) + \tilde{y}$ is solvable in the space X^* for every $\tilde{y} \in X^*$.

From Theorems 1–4 there follows the following

Alternative. Let $T, \widetilde{T}$ be completely continuous operators in X and X^*, respectively, with asymptotic derivatives B and B^*, respectively.

Then either the equations $x = T(x) + y,\ v = \widetilde{T}(v) + \tilde{y}$ are solvable for arbitrary $y \in X$, $\tilde{y} \in X^*$, or the equation $x = Bx$ has a nonzero solution. In this latter case (under the assumption that $[T(x) - Bx] \in \{\ker(I - B)^*\}^\perp$, $\forall x \in X$; $[\widetilde{T}(v) - B^*v] \in \{\ker(I - B)\}^\perp$, $\forall v \in X^*$) for the solvability of the equation $x = T(x) + y$ (respectively $v = \widetilde{T}(v) + \tilde{y}$) it is necessary and sufficient that $y \in \{\ker(I - B)^*\}^\perp$ (respectively $\tilde{y} \in \{\ker(I - B)\}^\perp$).

2. We now give an application of the results of Sec. 1 to nonlinear Urysohn integral equations. Such equations have been studied in many works; details concerning them are given in (^{2}). Let G be a bounded closed set of a finite-dimensional space. We require the following condition to be fulfilled.

Condition 1. The function $K(x, s, u)$, defined on $G \times G \times (-\infty, \infty)$, is continuous in u, measurable in (x, s), and satisfies the inequality

$$|K(x, s, u) - K_0(x, s)u| \leq a_1 + b_1 |u|^\gamma,$$

where the numbers $a_1 > 0,\ b_1 > 0,\ 0 < \gamma < 1$ do not depend on x, s, u; $K_0(x, s)$ is a function continuous on $G \times G$.

Let the number $p \geq 2,\ 1/p + 1/q = 1$, and let the functions $z_1(x), \dots, z_k(x)$ form a basis of the subspace (of $\mathcal{L}_q(G)$) of solutions of the equation

$$z(x) = \int_G K_0(t, x) z(t)\, dt.$$

Theorem 5. Suppose Condition 1 is fulfilled and the following condition holds:

$$\int_G \left\{ \int_G [K(x, s, u(s)) - K_0(x, s)u(s)]\, ds \right\} z_i(x)\, dx = 0, \qquad i = 1, \dots, k,$$

for all $u \in \mathcal{L}_p(G)$.

In order that, for a given $y \in \mathcal{L}_p(G)$, the Urysohn equation

$$u(x) = \int_G K[x, s, u(s)]\, ds + y(x)$$

be solvable in the space $\mathcal{L}_p(G)$, it is necessary and sufficient that the conditions

$$\int_G y(x) z_i(x)\, dx = 0, \qquad i = 1, \dots, k,$$

be fulfilled.

Theorem 6. Suppose Condition 1 is fulfilled and the equation

$$v(x) = \int_G K_0(x, t) v(t)\, dt$$

has in $\mathcal{L}_p(G)$ the unique solution $v(x) \equiv 0$.

Then the equation

$$u(x) = \int_G K[x, s, u(s)]\, ds + y(x)$$

is solvable in $\mathcal{L}_p(G)$ for every $y \in \mathcal{L}_p(G)$.

Analogous results are obtained with the aid of Theorems 3 and 4. We note that propositions analogous to Theorems 5 and 6 can be obtained when $1 < p < 2$, the function $K_0(x, s)$ is summable to some power, and the number a_1 is replaced by a summable function. Exactly the same results can also be proved for systems of Urysohn and Hammerstein integral equations.

3. We now consider the boundary-value problem for a nonlinear ordinary differential equation of order $2m$

$$(-1)^m \frac{d^m}{dx^m}\left[p_m(x)\frac{d^m u}{dx^m}\right] =$$

$$= \sum_{k=0}^{m-1}(-1)^k \frac{d^k}{dx^k}\left[T_k\left(x, u, \ldots, \frac{d^{m-1}}{dx^{m-1}}\right)\right] + z(x)$$

with homogeneous boundary conditions $u(a) = u'(a) = \ldots = u^{m-1}(a) = u(b) = u'(b) = \ldots = u^{m-1}(b) = 0$. We require that the following conditions be fulfilled.

Condition 2. a) All functions $T_k(x, u^0, u^1, \ldots, u^{m-1})$, $k = 0, 1, \ldots, m-1$, are continuous functions of the variable $x \in [a, b]$ and of the variables $u^0, u^1, \ldots, u^{m-1} \in (-\infty, \infty)$, satisfying the inequalities

$$\left|T_k(x, u^0, u^1, \ldots, u^{m-1}) - q_k(x)u^k\right| \le c_2 + c_3 \sum_{i=0}^{m-1} |u^i|^{\gamma_i},$$

$$\forall x \in [a, b], \qquad u^i \in (-\infty, \infty),$$

where the functions $q_k(x)$ are continuous on $[a, b]$, the constants $c_2 > 0$, $c_3 \ge 0$, $0 < \gamma_i < 1$;

b) the function $p_m(x)$, continuous on $[a, b]$, satisfies the inequality $p_m(x) \ge p_0 > 0$, $\forall x \in [a, b]$, $p_0 = \text{const}$.

Along with the nonlinear boundary-value problem, let us consider the linear boundary-value problem

$$(-1)^m \frac{d^m}{dx^m}\left[p_m(x)\frac{d^m u}{dx^m}\right] = \sum_{k=0}^{m-1}(-1)^k \frac{d^k}{dx^k}\left[q_k(x)\frac{d^k u}{dx^k}\right]$$

with boundary conditions $u(a) = \ldots = u^{m-1}(a) = u(b) = \ldots = u^{m-1}(b) = 0$.

We shall consider both boundary-value problems in the space $\dot{W}_2^m = \dot{W}_2^m(a, b)$ of S. L. Sobolev (^{4}). Let $\psi_1, \ldots, \psi_l$ be a basis of the subspace of solutions (in $\dot{W}_2^m$) of the linear boundary-value problem.

Theorem 7. *Let condition 2 be fulfilled and the equalities*

$$\sum_{k=1}^{m-1} \int_a^b \left\{ T_k \left(x, u, \frac{du}{dx}, \dots, \frac{d^{m-1}u}{dx^{m-1}} \right) - q_k(x) \frac{d^k u}{dx^k} \right\} \frac{d^k \psi_i}{dx^k}\, dx = 0, \qquad \forall u \in \dot{W}_2^m,$$

$$i = 1, \dots, l.$$

Then, for solvability of the nonlinear boundary-value problem in the space $\dot{W}_2^m$ *for* $z \in \mathcal{L}_2[a, b]$, *it is necessary and sufficient that the system of equalities be fulfilled*

$$\int_a^b z(x)\psi_i(x)\, dx = 0, \qquad i = 1, \dots, l.$$

Theorem 8. *Let condition 2 be fulfilled and let the linear boundary-value problem have in the space* $\dot{W}_2^m$ *only the zero solution.*

Then the nonlinear boundary-value problem is solvable in the space $\dot{W}_2^m$ *for any* $z \in \mathcal{L}_2[a, b]$.

Let us note that analogous results can be obtained for other boundary conditions, as well as for systems of differential equations. In doing so, the requirement of continuity of certain functions may be weakened.

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