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**Abstract**

**Full Text**

**V. A. TOPONOGOV**

**THEOREMS ON SHORTEST PATHS IN NON-COMPACT RIEMANNIAN SPACES OF POSITIVE CURVATURE**

*(Presented by Academician A. D. Aleksandrov on 18 VIII 1969)*

In the works of A. V. Pogorelov and W. Klingenberg, for simply connected compact Riemannian spaces  $V^m$  the following theorem was proved:

*If in a Riemannian space  $V^m$  the Riemannian curvature  $K$  at every point and in every two-dimensional direction does not exceed  $k_1 > 0$  and is positive for  $m = 2p$  (not less than  $k_1/4$  for  $m = 2p + 1$ ), then every arc of a geodesic line of length not greater than  $\pi/\sqrt{k_1}$  is a shortest path in  $V^m$  <sup>(2-4)</sup>.*

For noncompact Riemannian spaces an analogous theorem had been obtained only for spaces with a pole <sup>(5)</sup>.

In the present work arbitrary complete noncompact Riemannian spaces are considered.

Denote by  $R^m(R_0^m)$  a complete  $m$ -dimensional twice continuously differentiable Riemannian space whose curvature at every point and in every two-dimensional direction is positive (nonnegative).

**Theorem 1.** *If the Riemannian curvature of  $R^m$  at every point and in every two-dimensional direction does not exceed the number  $k_1 > 0$ , then every arc of a geodesic line whose length does not exceed  $\pi/\sqrt{k_1}$  is a shortest path in  $R^m$ .*

**Theorem 2.** *If the Riemannian curvature of  $R_0^m$  at every point and in every two-dimensional direction does not exceed some number  $k_1 > 0$ , then there exists a number  $\rho > 0$  such that every arc of a geodesic line of length not greater than  $\rho$  is a shortest path in  $R_0^m$ .*

In Theorem 2 the number  $\rho$  cannot be estimated in terms of  $k_1$ , as follows from the example of W. Klingenberg <sup>(1, p. 230)</sup>.

The proof of Theorems 1 and 2 rests essentially on the fundamental Lemmas 1 and 1<sup>0</sup>.

1. We shall call a **ray**  $p(A)$  in the space  $R^m(R_0^m)$  with initial point  $A$  a half-geodesic issuing from  $A$ , every arc of which is a shortest path in  $R^m(R_0^m)$ .

Since  $R^m(R_0^m)$  is a complete and noncompact space, through every point of  $R^m(R_0^m)$  there passes at least one ray. The distance between points  $A$  and  $B$

will be denoted by  $AB$ .

If  $ABC$  is a triangle in  $R^m(R_0^m)$  made up of shortest paths, then by  $A'B'C'$  we shall denote the triangle in the Euclidean plane whose sides are equal to the corresponding sides of the triangle  $ABC$ .

Let  $AB$  be a shortest path,  $p_1(A)$  a ray with initial point at  $A$ , and  $p_2(B)$  a ray with initial point at  $B$  in  $R^m$ . Denote by  $\alpha$  the angle between  $p_1(A)$  and the shortest path  $AB$ , and by  $\beta$  the angle between  $p_2(B)$  and the shortest path  $BA$ .

**Lemma 1.** *If on the ray  $p_1(A)$  in the space  $R^m$  there exists a sequence of points  $Q_n$  such that the shortest paths  $BQ_n$  converge to  $p_2(B)$ , and if neither of the rays  $p_1(A)$  and  $p_2(B)$  is a part of the other, then  $\alpha + \beta > \pi$ .*

**Proof.** Denote by  $\beta_n$  the angle in the triangle  $ABQ_n$  at the vertex  $B$ , and by  $\alpha'_n$  and  $\beta'_n$  the angles of the triangle  $A'B'Q'_n$  at the vertices  $A'$  and  $B'$ . Choose  $n_0$  so large that the triangle  $ABQ_{n_0}$  is a nondegenerate triangle. Take the ball  $T$  with center at the point  $A$  and radius

$$r = 2(AQ_{n_0} + AB + BQ_{n_0}).$$

Denote by  $k_0$  the minimum of the Riemannian curvature over all points of the ball  $T$  and over all two-dimensional directions. Apply Theorems 1 and 6 of [6] to the triangle  $ABQ_{n_0}$ ; it follows from them that, first, the perimeter of the triangle does not exceed  $2\pi/\sqrt{k_0}$ , and, second, the angle  $\alpha$  in the triangle  $AQ_{n_0}B$  is not less than the angle  $\alpha''_{n_0}$  in the triangle  $A''Q''_{n_0}B''$ , constructed on the two-dimensional sphere of radius  $1/\sqrt{k_0}$  with side lengths the same as those of the triangle  $AQ_{n_0}B$ , at the vertex  $A''$ . Comparing now the angle  $\alpha''_{n_0}$  of the triangle  $A''Q''_{n_0}B''$  with the angle  $\alpha'_n$  of the triangle  $A'B'Q'_n$ , we obtain

$$\alpha \geq \alpha_{n_0} + \delta, \tag{1}$$

where  $\delta$  is some positive number that can be estimated from below in terms of  $k_0$  and the side lengths of the triangle  $ABQ_{n_0}$ . By Theorem 5 of [6],

$$\alpha'_n \leq \alpha'_{n_0} \quad \text{for } n \geq n_0. \tag{2}$$

Moreover, by Theorem 6 of [6],

$$\beta_n \geq \beta'_n. \tag{3}$$

Finally,

$$\lim_{n \rightarrow \infty} (\alpha'_n + \beta'_n) = \pi. \tag{4}$$

From (1), (2), (3), and (4) it follows that

$$\alpha + \beta = \alpha + \lim_{n \rightarrow \infty} \beta_n \geq \lim_{n \rightarrow \infty} (\alpha'_n + \beta'_n) + \delta = \pi + \delta > \pi,$$

as was required to prove.

**Lemma 1<sup>0</sup>.** *If on the ray  $p_1(A)$  of the space  $R_0^m$  there exists a sequence of points  $Q_n$  such that the shortest paths  $BQ_n$  converge to  $p_2(B)$ , then  $\alpha + \beta \geq \pi$ .*

The proof of Lemma 1<sup>0</sup> is based on the same Theorems 1 and 5 of [6] and is much simpler than the proof of Lemma 1.

Lemma 2 follows from Lemma 1.

**Lemma 2.** *In  $R^m$  there exists no closed geodesic line.*

**2. Proof of Theorems 1 and 2.** Suppose that Theorems 1 and 2 are false. Then there exists a nondegenerate bigon  $\gamma$  of perimeter less than  $2\pi/\sqrt{k_1}$ . Let  $A$  be one of the vertices of  $\gamma$ . Denote by  $\gamma(A)$  a bigon of least length among all nondegenerate bigons with vertex at the point  $A$ . By the usual methods of the calculus of variations (see, for example, [4] or [7]) one can show that  $\gamma(A)$  is a geodesic loop with vertex at the point  $A$ . Let  $T(R)$  be the closed ball with center at the point  $A$  and radius  $R$ . Denote by  $\gamma(R)$  ( $\gamma_0(R)$ ) the geodesic loop in  $R^m$  ( $R_0^m$ ) of least length among all geodesic loops whose vertices lie in  $T(R)$ , by  $Q(R)$  ( $Q_0(R)$ ) the vertex of  $\gamma(R)$  ( $\gamma_0(R)$ ), and by  $f(R)$  ( $f_0(R)$ ) the length of  $\gamma(R)$  ( $\gamma_0(R)$ ).

By Lemma 2, the function  $f(R)$  is strictly monotone, while the function  $f_0(R)$  tends to zero as  $R \rightarrow \infty$ , by the assumption that Theorem 2 is false.

In both cases one can choose a sequence of numbers  $R_n \rightarrow \infty$  such that: 1) the shortest paths  $AQ(R_n)$  ( $AQ_0(R_n)$ ) converge to some ray  $p(A)$  ( $p_0(A)$ ); 2) the lower left derivatives of the function  $f(R)$  ( $f_0(R)$ ) at the points  $R_n$  are strictly negative.

We now choose  $n_0$  so large that the angle  $\alpha$  between  $p(A)$  ( $p_0(A)$ ) and  $AQ(R_{n_0})$  ( $AQ_0(R_{n_0})$ ) is less than  $\pi/4$ :

$$\alpha < \pi/4. \quad (5)$$

Divide the geodesic  $\gamma(R_{n_0})$  ( $\gamma_0(R_{n_0})$ ) by points  $A_1, A_2, \dots, A_{k+1} = Q(R_{n_0})$  in such a way that any segment  $A_i A_{i+1}$  ( $i = 1, \dots, k$ ) of the loop  $\gamma(R_{n_0})$  ( $\gamma_0(R_{n_0})$ ) is a shortest curve in  $R^m$  ( $R_0^m$ ). On the ray  $p(A)$  choose a sequence of points  $P_m$  such that, for any  $i$ , the shortest curves  $A_i P_m$  as  $m \rightarrow \infty$  converge to certain rays  $p_i(A_i)$  ( $p_i^0(A_i)$ ). Extend the rays  $p_i(A_i)$  ( $p_i^0(A_i)$ ) by geodesics  $\sigma_i$  ( $\sigma_i^0$ ) in the directions opposite to the directions of the rays  $p_i$  and  $p_i^0$ . On each geodesic  $\sigma_i$  ( $\sigma_i^0$ ), lay off from the points  $A_i$  segments of length  $\delta$ ; at the endpoints  $B_i(\delta)$  of these segments, as vertices, construct a polygon  $\bar{\gamma}(\delta)$  ( $\bar{\gamma}_0(\delta)$ ), whose perimeter we denote by  $p(\delta)$  ( $p_0(\delta)$ ).

By Lemma 1 (Lemma 1<sup>0</sup>), for sufficiently small  $\delta$ ,

$$p(\delta) < f(R_{n_0}) \quad (p_0(\delta) = f_0(R_{n_0}) + o(\delta)).$$

Next, by the usual methods of the calculus of variations one can show the existence of a geodesic loop  $\tilde{\gamma}(\delta)$  ( $\tilde{\gamma}_0(\delta)$ ) with vertex at the point  $B_1(\delta)$ , whose length does not exceed  $p(\delta)$  ( $p_0(\delta)$ ).

On the other hand, from Theorem 6 of [6] and (5) it follows that the angle between the shortest curve  $Q(R_{n_0})A$  and  $\sigma_1$  ( $Q_0(R_{n_0})$  and  $\sigma_1^0$ ) is less than  $\pi/4$ , and consequently, for sufficiently small  $\delta$ ,  $A_1B_1(\delta) < AA_1$ . Hence  $f(R) < f(R_{n_0})$  ( $f_0(R) \leq f_0(R_{n_0}) + o(\delta)$ ) for  $R_{n_0} - \delta \leq R < R_{n_0}$ , whence, in turn, it follows that the left lower derived number for the function  $f(R)$  ( $f_0(R)$ ) at the point  $R_{n_0}$  is not less than zero, contrary to the choice of the number  $n_0$ .

Thus Theorems 1 and 2 are proved.

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*Note: Figure translations are in progress. See original paper for figures.*

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