ENTROPY OF A SHIFT AND MARKOV MEASURES IN THE PATH SPACE OF A COUNTABLE GRAPH

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Abstract

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ENTROPY OF A SHIFT AND MARKOV MEASURES IN THE PATH SPACE OF A COUNTABLE GRAPH

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Let G be a directed graph with a countable set of vertices, which henceforth will be identified with the natural numbers. Denote the adjacency matrix of the graph G by $\Pi(G) = \Pi = (\pi_{ij})$. We shall assume that the graph is connected, i.e., any two of its vertices can be joined by a path. The length of a path is the number of edges in it. If the lengths of all closed paths (cycles) of the graph G are relatively prime, then the graph is called **aperiodic**. An infinite two-sided sequence of vertices $\{i_k,\ -\infty < k < \infty\}$ forms an infinite path if, for every k, there is an edge leading from i_k to i_{k+1}, i.e. $\pi_{i_k i_{k+1}} = 1$. Denote by $\Omega(G)$ the totality of all infinite paths of the graph G.

Let M be the set of numbers of the form $0, 1/n$ $(n = 1, 2, ...)$, and let $\Omega(M)$ be the set of infinite two-sided sequences of elements of M. If M is regarded as a compact subset of the line, then $\Omega(M)$ with the topology of the direct product will be a compact topological space. The mapping $n \mapsto 1/n$ induces an embedding of $\Omega(G)$ into $\Omega(M)$ and thereby defines a topology in $\Omega(G)$. The same topology can be obtained if one introduces the discrete topology on the set N of natural numbers and regards $\Omega(G)$ as a subset of the space $\Omega(N)$ of two-sided sequences of natural numbers with the topology of the direct product. Obviously, $\Omega(G)$ is closed in $\Omega(N)$ and not closed in $\Omega(M)$. Denote its closure in $\Omega(M)$ by $\overline{\Omega}(G)$.

In each of the spaces mentioned one can define the shift, which takes the sequence $\omega = \{\omega_k\}$ to $\omega' = \{\omega'_k\}$, where $\omega'_k = \omega_{k-1}$, and which is a homeomorphism. The shift $T(G)$ in the space $\Omega(G)$ is called a **topological Markov chain** (see ([1],[2])).

From the theorem proved in ([2]) it follows that the topological entropy (see ([3])) of the shift in the compact space $\overline{\Omega}(G)$ coincides with the upper bound of the metric entropies over all normalized invariant Borel measures concentrated on $\Omega(G)$.

Denote by $h(G)$ the common value of these two quantities and define $\lambda(G)$ by the condition $h(G) = \log \lambda(G)$ (the base of the logarithms is the same as in the definition of entropy). A measure μ in the space $\Omega(G)$ for which the metric entropy of the shift $h_\mu(T(G))$ is equal to $h(G)$ will be called a **measure with maximal entropy**. The example contained in (^{2}) shows that, in the case under consideration, unlike the case of a finite graph (see (^{4},^{5})), there need not exist a measure with maximal entropy. In this note we give necessary and sufficient conditions for the existence and uniqueness of such a measure.

1. Special graphs. First consider a graph G whose adjacency matrix $\Pi(G) = \Pi = (\pi_{ij})$ is determined by a nondecreasing sequence of natural numbers $\{k_s,\ s = 1, 2, ...\}$ according to the rule: $\pi_{ij} = 1$ if

$$i \neq \sum_{s=1}^{n} k_s, \quad j = i + 1$$

or

$$i = \sum_{s=1}^{n} k_s, \quad j = 1 + \sum_{s=1}^{m} k_s, \quad n \geq 1,\ m \geq 0$$

(we agree that $\sum_{s=1}^{0} k_s = 0$), and $\pi_{ij} = 0$ in all other cases. Graphs of the described—

graphs of this form will be called **special**. Such graphs, in connection with the problem of interest to us, were first considered by E. I. Dinaburg (^{6})*.

Theorem 1. *If G is a special graph defined by the sequence $\{k_s\}$, and $r(G) = r$ is the radius of convergence of the series*

$$\varphi(t) = \sum_{s=1}^{\infty} t^{k_s},$$

then $\lambda(G) = r^{-1}$ for $\varphi(r) \leq 1$ and $\lambda(G) = t_0^{-1}$ for $\varphi(r) > 1$, $\varphi(t_0) = 1$, $t_0 > 0$.

The proof is obtained by comparing the results of (^{2},^{6}).

Let us note that the shift in the path space of a special graph serves as a topological analogue of a special automorphism (see, for example, (^{7})), constructed from a Bernoulli automorphism with a countable number of states and a function depending only on the state at time zero and equal to k_n on the n-th state.

Theorem 2. *Let, in the notation of the preceding theorem, $r > 0$. Then, for the existence of a measure with maximal entropy, it is necessary and sufficient that one of the following two conditions hold:*

$$1)\ \varphi(r) > 1; \qquad 2)\ \varphi(r) = 1, \quad \varphi'(r) < \infty.$$

If a measure with maximal entropy exists, then it is unique, and after the introduction of this measure the shift $T(G)$ becomes a special automorphism constructed from a Bernoulli automorphism with states $E_1, E_2, \ldots$ and probabilities $p(E_n) = (\lambda(G))^{-k_n}$, $n = 1, 2, \ldots$, and the function $f(E_n) = k_n$.

In the proof the following simple

Lemma. *Let*

$$\psi(x) = \sum_{n=1}^{\infty} a_n x^n, \qquad \text{where } a_n \geq 0.$$

Then

$$x[\psi'(x)]^2 - \psi(x)\psi'(x) - x\psi(x)\psi''(x) \leq 0,$$

and equality is possible only in the case when $\psi(x) = cx^k$.

2. The general case. We now pass to the study of an arbitrary graph G, assuming only that $\lambda(G) < \infty$. From this assumption it follows that for any vertex i and any $n > 0$ there is only a finite number of cycles of length n passing through i (see (^{2}), Lemma 3). Fixing i, enumerate, in increasing order of their lengths, all cycles passing through i exactly once (cycles of the same length may be numbered in an arbitrary order). Let $k_s(i)$ be the length of the s-th cycle in our enumeration, and let $G(i)$ be the special graph defined by the sequence $\{k_s(i)\}$. We shall call $G(i)$ the **special representation** of the graph G corresponding to the vertex i.

Using the special representation and the results of Section 1, one can establish the following facts:

Theorem 3. *Let G be a connected graph and $\lambda(G) < \infty$. Then: 1) if a measure with maximal entropy exists, then it is unique; 2) if i is an arbitrary vertex of the graph G and $G(i)$ is the special representation corresponding to it, then $T(G)$ and $T(G(i))$ possess measures with maximal entropy simultaneously; 3) if μ_0 is a measure with maximal entropy, then μ_0 is a Markov measure positive on every open set (the latter means that the random variables $x_n(\omega) = \omega_n$, $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Omega(G)$, form a Markov chain).*

Corollary. *If G is a connected aperiodic graph, $\lambda(G) < \infty$, and μ_0 is a measure with maximal entropy, then $(T(G), \mu_0)$ is a K-automorphism.*

Let us give one more fact, useful for applications and following from the results of (^{2}).

Theorem 4. *If G is a connected graph, then for every $c > 0$ there exists an ergodic Markov measure, positive on every open set,* measure μ_c such that $h_\mu(T(G)) > h(T(G)) - c$ when $\lambda(G) < \infty$, and $h_{\mu_c}(T(G)) > c$ when $\lambda(G) = \infty$.

3. Relation with the matrix $\Pi(G)$. As in the case of a finite graph G (see (^{4},^{5})), the entropy $h(G)$ and the measure with maximal entropy, if it exists, can be directly related to the matrix $\Pi(G)$. Suppose that $\lambda(G) < \infty$. Then all powers $\Pi^n = (\pi_{ij}^{(n)})$, $n > 0$, of the matrix $\Pi = \Pi(G)$ are defined (see (^{2}), Lemma 3), and the radius of convergence $R(G)$ of the series

$$\sum_n \pi_{ij}^{(n)} t^n$$

does not depend on i and j (see (^{8})).

Theorem 5. *If G is a connected graph, $\lambda(G) < \infty$, and $R = R(G)$, then*

$$h(G) = \log \lambda(G) = -\log R.$$

The number $R(G)$ is called the convergence parameter of the matrix $\Pi(G)$, and $[R(G)]^{-1}$ serves as an analogue of the maximal eigenvalue (see (^{9})). If $R(G) = R$ and $\pi_{ij}^n R^n \to 0$, i.e. the matrix $\Pi(G)$ is, by definition, R-recurrent and R-positive, then R^{-1} is a genuine eigenvalue: there exist sequences of positive numbers $x(G) = x = (x_1, x_2, ...)$, $y(G) = y = (y_1, y_2, ...)$, such that

$$\sum_j \pi_{ij} x_j = (1/R) x_i,$$

$$\sum_i \pi_{ij} y_i = (1/R) y_j,$$

and the vectors x, y are determined uniquely up to a factor, and

$$\sum_i x_i y_i < \infty$$

(see (^{9})).

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* Taking the occasion, I note at the request of the author of (^{6}) that, for the validity of assertion 4) from that paper, the condition $f(k) > c \ln k$ must be replaced by $f(k) < c(\ln k)^a$, $a > 1$.

Theorem 6. *If G is a connected graph and $\lambda(G) = 1/R < \infty$, then for the existence of a measure μ_0 with maximal entropy it is necessary and sufficient that the matrix $\Pi(G)$ be R-recurrent and R-positive. Under these conditions μ_0 is the Markov measure corresponding to transition probabilities*

$$p_{ij} = x_j \pi_{ij} / \lambda(G) x_i, \quad i, j = 1, 2, \dots,$$

and stationary probabilities

$$p_i = x_i y_i,$$

where the vectors $x(G) = x,\ y(G) = y$ are normalized so that

$$\sum_i x_i y_i = 1.$$

4. Application to Y-systems. The results of § 2 find application in the theory of classical dynamical systems. In particular, with their help the following can be proved.

Theorem 7. *Let T be a C^2-diffeomorphism of a closed n-dimensional Riemannian manifold M, possessing an integral invariant* and satisfying the condition Y of D. V. Anosov ([10]). *Then on M there exists a unique invariant, relatively T-normalized, Borel measure μ_0, for which the metric entropy $h_{\mu_0}(T)$ assumes the maximum of the possible values, equal to the topological entropy (see ([3],[11])). The transformation T of the space M with measure μ_0 is a K-automorphism.*

Remark. The existence of a measure with maximal entropy for a Y-diffeomorphism was earlier proved by R. Bowen ([12]).

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* In the presence of an integral invariant, each leaf of the expanding and contracting foliations corresponding to T is everywhere dense in M (see $(^{10})$). Only this is needed for the proof of the last assertion of Theorem 7.

Note: Figure translations are in progress. See original paper for figures.

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