STUDIES ON MULTIDIMENSIONAL LIMIT THEOREMS OF PROBABILITY THEORY

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Abstract

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STUDIES ON MULTIDIMENSIONAL LIMIT THEOREMS OF PROBABILITY THEORY

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We consider the asymptotic behavior of the density $p_n(\mathbf{x})$ and of the probability function $P_n(A)$ of the sum

$$S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (\xi_j - M\xi_j)$$

of independent identically distributed k-dimensional random vectors $\xi_j = (\xi_{1j}, \xi_{2j}, \dots, \xi_{kj})$, $j = 1, 2, \dots, n$, with nondegenerate covariance matrix V, where $M\xi_j$ is the vector of mathematical expectations.

The present paper is a continuation of the author' s investigations (^{1,2}) and refines a number of known ($^{3-12}$) multidimensional limit theorems.

1. In order to reveal the structure of the remainder term in the asymptotic expansions for $P_n(A)$ and $p_n(\mathbf{x})$, an upper estimate has been obtained for the modulus of the characteristic function $f(\mathbf{t})$ of the random vector ξ_1, whose distribution function has a density $q(\mathbf{x})$ bounded by a constant C. It should be noted that S. M. Sadikova obtained (^{6}) sufficient conditions (certain restrictions on the density $q(\mathbf{x})$) for the power decrease of the modulus of the characteristic function $f(\mathbf{t})$ as $\|\mathbf{t}\| \to \infty$, where $\|\mathbf{t}\|$ is the length of the vector $\mathbf{t} \in R^k$. We have obtained the following lemma, from which for $k = 1$ the known (^{13}) result for one-dimensional characteristic functions follows.

Lemma 1. For all $\mathbf{t} \in R^k$ the inequality holds

$$|f(\mathbf{t})| \leq \exp\left\{-\frac{\pi^2}{27} \frac{(\mathbf{t}V\mathbf{t}')((k-1)!!)^2}{C^2|V|(8\pi)^k k^{k-1}\left(2\pi + \sqrt{k}\sqrt{\mathbf{t}V\mathbf{t}'}\right)^2}\right\}. \qquad (1)$$

Here $\mathbf{t}V\mathbf{t}'$ is the quadratic form corresponding to the matrix V; $|V|$ is the determinant of the matrix V.

From formula (10) of paper (^{2}) and inequality (1) it follows that

Theorem 1. If the distribution function of the random vector ξ_1 has a density bounded by a constant C, and finite moments of order s ($s \geq 3$), then for all $\mathbf{x} \in R^k$

$$\left|p_n(\mathbf{x}) - \sum_{j=0}^{s-3}\left(\frac{1}{\sqrt{n}}\right)^j P_j(-\varphi)(\mathbf{x})\right| \leq \frac{2^{s-1}\beta_s\Gamma((k+s)/2)}{\pi^{k/2}\sqrt{|V|}\,\Gamma(k/2)n^{(s-2)/2}} +$$

$$+ Cn^{k/2}\exp\left\{-\frac{\pi^2}{27}\,\frac{(n-2)((k-1)!!)^2}{C^2\sqrt{|V|}(8\pi)^k k^{k-1}\left(16\pi\beta_s^{1/(s-2)} + \sqrt{k}\right)^2}\right\}.$$

Here $P_j(-\varphi)(\mathbf{x})$ are known functions of many variables,

$$\beta_s = \sup_{\|\mathbf{t}\|=1}\frac{M|(\xi_1 - M\xi_1, \mathbf{t})|^s}{\left(M(\xi_1 - M\xi_1, \mathbf{t})^2\right)^{s/2}}, \qquad \Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du,$$

$\|\mathbf{t}\|$ is the modulus of the vector $\mathbf{t} \in R^k$, and $(\xi_1, \mathbf{t})$ is the scalar product.

In the proof of Theorem 1 we shall use the following lemma.

Lemma 2. If ξ_1 has finite moments of order s ($s \geqslant 3$), then, for $\left(M|(\xi_1 - M\xi_1, t)|^s/tVt'\right)^{1/(s-2)} \leqslant \sqrt{n}/8$,

$$\left|f^n\left(\frac{t}{\sqrt{n}}\right) - \exp\left(\frac{tVt'}{2}\right)\left(1 + \sum_{j=1}^{s-3}\left(\frac{1}{\sqrt{n}}\right)^j P_j(it)\right)\right| \leqslant \frac{2^{3s/2}M|(\xi_1 - M\xi_1, t)|^s}{n^{(s-2)/2}}\exp\left(-\frac{tVt'}{4}\right);$$

$P_j(it)$ are known polynomials.

2. Let us denote by $\mathfrak{N}$ and $\mathfrak{A}$, respectively, the classes of Borel and convex sets A in R^k; $\Phi(\mathbf{x})$ is the k-dimensional normal distribution with parameters $(0, V)$, where 0 is the zero vector in R^k;

$$G_{s-2}(A) = \sum_{j=0}^{s-2}\left(\frac{1}{\sqrt{n}}\right)^j \int_A P_j(-\varphi)(\mathbf{y})\,d\mathbf{y};$$

$C_1, C_2, \ldots$ are positive constants not depending on the set A or on the number of summands n in the sum S_n; $\Phi_T(\mathbf{x})$ is the k-dimensional distribution function with characteristic function $\exp\{-\|t\|^2/T\}$, $T > 0$; $H_T(\mathbf{x})$ is the distribution function with characteristic function $h(t/T)$, where

$$h(\mathbf{t}) = \int_{R^k} e^{i(\mathbf{t},\mathbf{x})} \prod_{j=1}^{k} \frac{96}{\pi} \left(\frac{\sin x_j/4}{x_j} \right)^4 d\mathbf{x};$$

$(\mathbf{t}, \mathbf{x})$ is the scalar product of the vectors $\mathbf{t}, \mathbf{x} \in R^k$.

By the method of truncating random vectors (see (2)) the following theorems are proved.

Theorem 2. If ξ_1 has finite moments of order s $(s \geqslant 3)$ and Cramér' s condition is satisfied,

$$\lim_{\|t\| \to \infty} |f(t)| < 1, \tag{c}$$

then, uniformly in $A \in \mathfrak{A}$, the relation

$$|(P_n - G_{s-2}) * \Phi_T(A)| \leqslant d_1(n)/n^{(s-2)/2}$$

holds for $T = (C_1 n)^{(s-1)/2}$. Here and below $\lim d_i(n) = 0$, $i = 1, 2, \ldots$; $*$ denotes convolution.

Theorem 3. If ξ_1 has finite moments of order $2+\delta$ $(0 < \delta \leqslant 1)$, then, uniformly in $A \in \mathfrak{A}$,

$$|(P_n - \Phi) * H_T(A)| \leqslant C_2/n^{\delta/2}$$

for $T = (C_3 n)^{\delta/2}$.

Theorem 4. Suppose that the distribution function of the random vector ξ_1 is non-lattice, i.e. $|f(t)| = 1$ only at the single finite point $t = 0$, and has finite moments of third order; then, uniformly in $A \in \mathfrak{A}$,

$$|(P_n - G_1) * H_T(A)| \leqslant d_2(n)/\sqrt{n}$$

for $T = \lambda(n)\sqrt{n}$, where $\lim_{n\to\infty} \lambda(n) = \infty$.

With the aid of the remarkable so-called "inversion formula" of B. Bahr (10), under widely accepted natural conditions, for broad classes of sets A there are obtained (Theorems $2'$-$4'$) asymptotic expansions for the probability function $P_n(A)$. The estimates obtained for the remainder terms are optimal with respect to n and sharper than the known ones (see (4-6, 10, 12)).

Theorem $2'$. Under the conditions of Theorem 2 we have

$$|P_n(A) - G_{s-2}(A)| \leqslant \frac{d_1(n)}{n^{(s-2)/2}} + 2\alpha \left(A_1 (C_1 n)^{-(s-1)/2} \right)$$

for $A \in \mathfrak{A}$ and

$$\sup_{A \in \mathfrak{A}} |P_n(A) - G_{s-2}(A)| = o\left(n^{-(s-2)/2}\right). \tag{2}$$

Theorem 3′. Under the conditions of Theorem 3 we have

$$|P_n(A) - \Phi(A)| \leq C_2 n^{-\delta/2} + 2\alpha\left(A_1 (C_3 n)^{-\delta/2}\right)$$

for $A \in \mathfrak{A}$, and

$$\sup_{A \in \mathfrak{A}} |P_n(A) - \Phi(A)| \leq C_4 n^{-\delta/2}.$$

Theorem 4′. Under the conditions of Theorem 4 we have

$$|P_n(A) - G_1(A)| \leq d_2(n)/\sqrt{n} + 2\alpha\left(A,\, 1/\lambda(n)\sqrt{n}\right)$$

for $A \in \mathfrak{A}$, and

$$\sup_{A \cap \mathfrak{A}} |P_n(A) - G_1(A)| = o(1/\sqrt{n}).$$

Here

$$\alpha(A, \varepsilon) = \sup_{y \in R^k} \int_{(A)_\varepsilon} |dQ(\mathbf{x} + \mathbf{y})|,$$

where $(A)_\varepsilon$ is the ε-neighborhood of the set A.

In Theorems 2′, 3′, and 4′, respectively, $Q(\mathbf{x}) = G_{s-2}(\mathbf{x})$, $\Phi(\mathbf{x})$, and $G_1(\mathbf{x})$. Theorems 2′–4′ are new also for one-dimensional probability functions $P_n(A)$.

It should be noted that for the class of k-dimensional "intervals" relation (2) was obtained by different methods by the author in ([2]) and by B. Bahr in ([10]).

3. Finally, by the method of truncating random vectors from the local limit theorem we obtain the following asymptotic expansion, which includes the known result of R. R. Rao ([5]).

Theorem 5. Let the random vector ξ_1 take values from the k-dimensional lattice

$$\{\mathbf{a} + \mathbf{v}H : \mathbf{v} = (v_1, v_2, \dots, v_k),\ v_1, v_2, \dots, v_k = 0, \pm 1, \pm 2, \dots\}$$

with maximal "span of the distribution" $h = |\det H|$, and have finite moments of order s ($s \geq 3$); then, uniformly in $A \in \mathfrak{A}$,

$$P_n(A) = \int d \prod_{l=1}^{k} \left\{ \sum_{\nu=0}^{s-2} \frac{(-1)^\nu}{\nu!} \left(\frac{h}{\sqrt{n}\, A_{ll}} \right)^\nu B_\nu \left(\left(\frac{\sqrt{n}\,\mathbf{y} + nM\xi_1 - n\mathbf{a}}{n}, \mathbf{q}_l \right) \right) \times \frac{\partial^\nu}{\partial y_l^\nu} \right\} G(\mathbf{y}) + o\left(n^{-(s-2)/2}\right). \tag{3}$$

Here the vector $\mathbf{q}_l$ has coordinates A_{lj}, $j = 1, 2, \ldots, k$; A_{lj} is the algebraic cofactor of the element with indices l, j in the determinant of the transposed matrix H'; $B_0(x) \equiv 1$,

$$-\frac{B_\nu(x)}{\nu!} = \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i m x}}{(2\pi i m)^\nu}, \qquad \nu = 2, 3, \ldots$$

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