ISOMETRIC EMBEDDINGS AND IMMERSIONS

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ISOMETRIC EMBEDDINGS AND IMMERSIONS

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I. Introduction. In the present note theorems are formulated that describe the structure of the space of isometric immersions of one pseudo-Riemannian (in particular, Riemannian) manifold into another. These theorems generalize and refine the results of Nash and Kuiper on isometric C^1-immersions, Nash' s theorems on isometric embeddings of the classes C^∞, C^a (analytic), the local theorem of Janet–Cartan–Burstin, and also the results of paper ([5]), which, in particular, contains a survey of the question and the literature. From the theorems of Sec. III of the present work there follow the immersion theorems of Smale–Hirsch and their generalizations (see ([2])). The propositions of Sec. VII are adjacent to paper ([4]).

II. Auxiliary notions. If η, θ are real vector bundles over A with fibers η_a, θ_a, $a \in A$, then a homomorphism (monomorphism) $\eta \to \theta$ is called a continuous family of linear (linear one-to-one) mappings $\eta_a \to \theta_a$. If ξ is a bundle over B and $f : A \to B$ is a continuous mapping, then the bundle induced over A is denoted by $f^*(\xi)$. A morphism (injection) of the bundle η over A into the bundle ξ over B is a pair (f, φ), where $f : A \to B$ is a continuous mapping and $\varphi : \eta \to f^*(\xi)$ is a homomorphism (monomorphism). For a morphism $\psi = (f, \varphi)$ we put $|\psi| = \varphi$. A form g in the bundle η over A is a continuous family of quadratic forms g_a, $a \in A$, in the fibers η_a. If the form g is nondegenerate, i.e. is reducible in each fiber to the form

$$\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2$$

with $n = \dim \eta$, then it is called a metric; by $\operatorname{pos} g$ we denote the rank k of the positive part, and by $\operatorname{neg} g$ the rank $n - k$ of the negative part of the metric g. A nondegenerate form g with $\operatorname{neg} g = 0$ is called positive.

In what follows W denotes a manifold with tangent bundle ω and metric h in ω (i.e. W is a pseudo-Riemannian manifold with metric h). If $f : A \to W$ is a continuous mapping, then by $f^*(h)$ we denote the metric induced in the bundle $f^*(\omega)$; for a homomorphism φ of the bundle η over A into the bundle $f^*(\omega)$, by

$g(\varphi)$ we denote the form in η induced by it (from the metric $f^*(h)$); and for a morphism $\psi = (f, \varphi) : \eta \to \omega$ we put $g(\psi) = g(\varphi)$.

In the sequel V denotes a manifold with a distribution α (a distribution α on V is a subbundle α of the tangent bundle $\tau(V)$, i.e. a field of tangent planes on V) and a metric g in α. A homomorphism φ of the bundle α into the bundle $f^*(\omega)$ induced by a continuous mapping $f : V \to W$ is called isometric if $g(\varphi) = g$. Analogously, a morphism $\psi : \alpha \to \omega$ is called isometric if $g(\psi) = g$. If $f : V \to W$ is a smooth mapping, then by $\delta_f : \alpha \to \omega$ we denote the restriction to α of the differential $d_f : \tau(V) \to \omega$. A mapping $f : V \to W$ of class C^r, $r = 1, 2, \dots, \infty, a$, will be called a C^r-isometry if the morphism δ_f is isometric. A C^1-mapping $f : V \to W$ will be called short if the form $g - g(\delta_f)$ is positive.

In this work the manifolds, bundles, and metrics are assumed to be C^∞-smooth, and the notion of approximation is used with respect to the space $C^r(V, W)$, $0 \le r \le \infty$, endowed with the fine C^r-topology (see ([6])).

A C^1-isometry $V \to W$ will be called **non-loose** if it admits a C^1-approximation by short mappings $V \to W$. For example, if $W = R^q$, and the manifold V is compact, or if $\operatorname{neg} h > \operatorname{neg} g$, then every isometry $V \to W$ is non-loose, while the standard isometry $R^{q-k} \to R^q$ is not non-loose.

III. C^1-isometries.

Theorem 1. *Let $f : V \to W$ be a smooth mapping and suppose that one of the following two conditions is satisfied:*

a) $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} h > \operatorname{neg} g$,

b) $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} g = 0$ and f is a short mapping.

Then, for the existence of a C^0-approximation of the mapping f by non-loose C^1-isometries $V \to W$, it is necessary and sufficient that there exist an isometric homomorphism $\alpha \to f^*(\omega)$.

Corollary. A. *If the manifold V is contractible,* $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} h > \operatorname{neg} g$, *then every continuous mapping $V \to W$ is approximated by C^1-isometries.*

A manifold M is called a π-manifold if the tangent bundle of the manifold $M \times R^1$ is trivial; an example of a π-manifold is the n-dimensional sphere.

B. *A short C^1-mapping of an n-dimensional Riemannian π-manifold into R^{n+1} admits a C^0-approximation by C^1-isometries.* (We note that a short mapping is not assumed to be an immersion.)

C. *For any linearly independent vector fields $X_1, \dots, X_n$ on a Riemannian manifold M, there exist C^1-functions $f_1, \dots, f_{n+1} : M \to R^1$ such that*

$$\sum_{i=1}^{n+1} (X_k f_i)(X_l f_i) = (X_k, X_l), \quad 1 \le k,\ l \le n.$$

(Here Xf denotes the derivative of the function f along X (see (1)), and (X, Y) denotes the scalar product of the fields X and Y.)

Theorem 2. *The mapping from the space of C^1-isometries $V \to W$ to the space of isometric morphisms $\alpha \to \omega$, which assigns to an isometry f the morphism δ_f, is a weak homotopy equivalence (i.e., induces an isomorphism of homotopy groups) in the following two cases:*

a) $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} h > \operatorname{neg} g$;

b) $W = R^q$ with $q > \operatorname{pos} g$.

Corollary. *Two homotopic C^1-isometries $V \to W$ with $\operatorname{pos} h > \operatorname{pos} g + \dim V$, $\operatorname{neg} h > \operatorname{neg} g + \dim V$ can be joined by a path in the space of C^1-isometries $V \to W$.*

IV. **C^1-embeddings.** Two monomorphisms $\eta \to \theta$ are called homotopic if they are connected by a continuous family of monomorphisms $\eta \to \theta$.

Theorem 3. *Suppose that $\alpha = \tau(V)$ (i.e., V is a pseudo-Riemannian manifold with metric g). Let $f : V \to W$ be a smooth embedding and suppose that one of the following two conditions is satisfied:*

a) $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} h > \operatorname{neg} g$,

b) $\operatorname{pos} h > \operatorname{pos} g$, $\operatorname{neg} g = 0$ and f is a short mapping.

Then, for the existence of a C^0-approximation of the mapping f by non-loose isometric C^1-embeddings $V \to W$, diffeotopic to the embedding f, it is necessary and sufficient that there exist an isometric monomorphism $\alpha \to f^*(\omega)$ homotopic to the monomorphism $|\delta_f| = |d_f| : \alpha \to f^*(\omega)$.

Corollary. *If V is compact, then every continuous mapping $V \to W$ with $\operatorname{pos} h > \operatorname{pos} g + \dim V$, $\operatorname{neg} h \geq \operatorname{neg} g + \dim V$ is approximated by isometric C^1-embeddings.*

V. **Free immersions.** For a bundle η over A, by η^2 we denote its symmetric square. By $X \circ Y \in (\eta^2)_a$, $a \in A$, we denote the vector corresponding to the pair of vectors $X, Y \in \eta_a$. (The symmetric square η^2 is the bundle associated with η, whose fiber $(\eta^2)_a$, $a \in A$, is the symmetric product $\eta_a \circ \eta_a$; see (2).)

Further, the distribution α on V is assumed to be involutive. (A distribution α is called involutive (see (4)) if the Poisson bracket of two vector fields belonging to α also belongs to α.)

Consider a C^2 mapping $f : V \to W$. Suppose that the form $g(|\delta_f|) = g(\delta_f)$ is nondegenerate, and denote by $P : f^*(\omega) \to f^*(\omega)$ the projection (orthogonal with respect to the metric $f^*(h)$) onto the orthogonal complement of the image of the bundle α under the homomorphism $|\delta_f| : \alpha \to f^*(\omega)$. Define the homomorphism $|\delta_f|^2 : \alpha^2 \to f^*(\omega)$ by the following condition. If $X, Y \in \alpha_v$, $v \in V$, $\overline{Y}$ is a smooth section of the bundle α extending the vector Y, and Y_1 is the image of the section $\overline{Y}$ under the homomorphism $|\delta_f|$, then

$$|\delta_f|^2(X \circ Y) = P\nabla_{XY}1,$$

where by $\nabla_{XY}1 \in (f^*(\omega))_v$ is denoted the covariant derivative along X of the section Y^0 of the bundle $f^*(\omega)$ with the connection induced from the pseudo-Riemannian connection in ω. A C^2 mapping $f : V \to W$ will be called α-free if the form $g(\delta_f)$ induced in α is nondegenerate, and the form $g(|\delta_f|^2)$ induced in α^2 is positive. If $\alpha = \tau(V)$, then an α-free mapping is called free. For example, an immersion $R^1 \to W$ is free if it induces in R^1 a nondegenerate form, its geodesic curvature does not vanish, and on the principal normal the form h is positive.

Theorem 4. *If the manifolds V, W, the distribution α, and the metrics g, h are analytic, then an α-free C^∞-isometry $V \to W$ admits a C^∞-approximation by analytic isometries $V \to W$.*

Next put $n = \dim \alpha$, $s = s_n = \dim \alpha^2 = n(n+1)/2$.

Theorem 5. *In order that a given C^1-isometry $f : V \to W$ admit a C^1-approximation by α-free C^∞-isometries $V \to W$ for*

$$\operatorname{pos} h \geq s + \operatorname{pos} g + 2n + 5,$$

it is necessary and sufficient that there exist a monomorphism $\varphi : \alpha^2 \to f^*(\omega)$ whose image is orthogonal (with respect to the metric $f^*(h)$) to the image of the homomorphism $|\delta_f| : \alpha \to f^*(\omega)$ and for which the form $g(\varphi)$ is positive.

Combining Theorems 3, 4, 5 leads to the corollaries:

Corollary. A. *For*

$$\operatorname{neg} h \geq \operatorname{neg} g + \dim V, \qquad \operatorname{pos} h \geq s + \dim V + 2n + 5$$

every continuous mapping $V \to W$ admits a C^0-approximation by α-free C^∞-isometries $V \to W$.

B. *An n-dimensional Riemannian manifold of class C^i, $i = \infty, a$, has an isometric C^i-embedding in R^q with $q = s + 3n + 5$, and for a complete manifold the embedding may be chosen without a limit set; if the manifold is compact and analytic, then it admits an isometric C^a-embedding in R^q with $q = s + 3n + 4$.*

Call a morphism $\varphi : \alpha \oplus \alpha^2 \to \omega$ (where $\alpha \oplus \alpha^2$ denotes the Whitney sum of the bundles α and α^2) semi-isometric if the form $g(\varphi)$ induced by it in $\alpha \oplus \alpha^2$ is nondegenerate, coincides on the subbundle α with g, and

$$\operatorname{neg} g(\varphi) = \operatorname{neg} g.$$

Theorem 6. *The mapping of the space of α-free C^∞-isometries $V \to W$ into the space of semi-isometric morphisms $\alpha \oplus \alpha^2 \to \omega$, which assigns to an isometry f the morphism*

$$(f, |\delta_f| \oplus |\delta_f|^2) : \alpha \oplus \alpha^2 \to \omega$$

(where $|\delta_f| \oplus |\delta_f|^2 : \alpha \oplus \alpha^2 \to f^*(\omega)$ denotes the direct sum of the homomorphisms $|\delta_f|$ and $|\delta_f|^2$), is a weak homotopy equivalence in the following two cases:

$$\text{a)} \quad \operatorname{pos} h \geq s + \operatorname{pos} g + 2n + 5, \quad \operatorname{neg} h > \operatorname{neg} g;$$
$$\text{b)} \quad W = R^q, \quad q \geq s + 3n + 5.$$

Corollary. *Two free isometric immersions of an n-dimensional Riemannian manifold in R^q with*

$$q \geq s + 3n + 5$$

are connected by a C^∞ bending.

VI. Riemannian manifolds. The aim of what follows is to remove, in a number of cases, the dimensional restrictions of the preceding section.

Theorem 7. *Let V be an n-dimensional Riemannian manifold, V_0 its connected closed submanifold, and W a contractible Riemannian manifold. Denote by β the restriction to V_0 of the tangent bundle $\tau(V)$, and by ν the normal bundle of the submanifold V_0.*

a) If ν is a trivial one-dimensional bundle, and the manifold V_0, as a Riemannian manifold with the metric induced from V, is reducible, or if $\dim \nu \geq 2$, then, in order that some neighborhood $U \subset V$ containing the submanifold V_0 have a free isometric embedding in W, it is necessary and sufficient that there exist an injection $\beta \oplus \beta^2 \to \tau(W)$. (For a π-manifold V an injection $\beta \oplus \beta^2 \to \tau(W)$ exists when $\dim W \geq s + n$.)

b) If the manifolds V, W and the submanifold V_0 are analytic, the bundle β is trivial, and the bundle ν has a nonvanishing section and $\dim \nu \geq 3$, then for $\dim W \geq s = n(n+1)/2$ some neighborhood $U \subset V$ containing the submanifold V_0 has an analytic isometric embedding in W.

Theorem 8. Let V_0 be a compact $(n-1)$-dimensional Riemannian π-manifold, and let $f_1, f_2 : V_0 \to R^q$ be free isometric C^∞-immersions (C^∞-embeddings). If $q \geq s + 2n + 2$ ($s = n(n+1)/2$), then for some $l > 0$ there exists a free isometric C^∞-immersion (C^∞-embedding) f of the cylinder $V_0 \times [0, l]$ into R^q, such that $f|_{V_0 \times 0} = f_1$, $f|_{V_0 \times l} = f_2$.

VII. **Topological propositions.** A. Let M be a smooth open π-manifold, W a Riemannian manifold, $f : M \to W$ a continuous map, and k a real number. For the existence of a free C^∞-immersion $M \to W$ homotopic to the map f and inducing on M a metric of constant curvature k, it is necessary and sufficient that there exist a monomorphism $\tau(M) \oplus (\tau(M))^2 \to f^*(\tau(W))$.

B. Let M be a smooth n-dimensional π-manifold and let $f, g : M \to R^q$ be continuous maps. If $q > s/2 + n$ ($s = n(n+1)/2$), then f, g admit C^0-approximation by C^∞-immersions $f_1, g_1 : M \to R^q$ inducing on M one and the same metric.

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Note: Figure translations are in progress. See original paper for figures.