ON THE DENSITIES OF MEASURES CORRESPONDING TO SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS WITH RANDOM FUNCTIONS

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Abstract

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A. D. Shatashvili

ON THE DENSITIES OF MEASURES CORRESPONDING TO SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS WITH RANDOM FUNCTIONS

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Finding the densities of measures corresponding to random processes is one of the important problems in the theory of random processes (see, on this subject, works ([1−10])), and the densities themselves are used in the solution of many problems in the theory of random processes: nonlinear extrapolation and filtering, optimal control, problems of mathematical statistics, and computation of distributions of various functionals.

In the present article we consider random processes $x_1(t)$ and $x_2(t)$ with values in the m-dimensional Euclidean space E_m, which are solutions of the differential equations

$$dx_2(t)/dt + L(t)x_2(t) + f(t, x_2(t)) = \xi(t), \qquad (0 \le t \le T),$$
$$x_2(0) = \xi(0) = 0; \tag{1}$$

$$dx_1(t)/dt + L(t)x_1(t) = \xi(t) \qquad (0 \le t \le T),$$
$$x_1(0) = \xi(0) = 0, \tag{2}$$

where $\xi(t)$ is a Gaussian process defined on the interval $[0, T]$ with values in E_m, with zero mathematical expectation, $M\xi(t) = 0$, and continuous correlation matrix $R(t, s)$ in the domain $[0, T] \times [0, T]$; the function $f(t, x)$ is defined and continuous jointly in both variables in the domain $[0, T] \times E_m$, takes its values in E_m, and

$$\sum_{j=1}^{m} \left\| \frac{\partial f(t, x)}{\partial x_j} \right\| < \infty \qquad (t \in [0, T],\, x \in E_m) \tag{3}$$

(here and below the symbol $\|\cdot\|$ denotes the norm in E_m, and the symbol $(\cdot,\cdot)$ denotes the scalar product in E_m); and $L(t)$ is a linear operator continuously depending on t and acting in E_m.

Let μ_1 and μ_2 be the measures generated respectively by the solutions of the differential equations (2) and (1) on the minimal σ-algebra that contains all cylindrical sets of the space of all vector functions defined on the interval $[0, T]$ and taking their values in the Euclidean space E_m.

In the present article conditions are established under which the measure μ_2 is absolutely continuous with respect to the measure μ_1, and the corresponding density is written out explicitly. We note that the process $x_1(t)$ is Gaussian, and its correlation matrix $B(t,s)$ is simply expressed in terms of the correlation matrix $R(t,s)$ and the fundamental matrix of solutions $Y(t)$ of the homogeneous equation

$$dY(t)/dt - Y(t)L(t) \equiv 0, \qquad Y(0) = I \tag{4}$$

(I is the identity matrix):

$$B(t,s) = Y(t)R(t,s)Y_*(s), \tag{5}$$

where $Y_*(t)$ denotes the matrix transposed to $Y(s)$.

Denote by $\mathfrak{F}_t$ the σ-algebra generated by the quantities $\xi(s)$ for $s \leqslant t$, and suppose that the following conditions are satisfied:

1) There exists a matrix function $Q(t,s)$ such that

$$B(t,s) = \int_0^T Q(t,u)Q(s,u)\,du. \tag{6}$$

2) There exists an $\mathfrak{F}_t$-measurable Wiener process $w(t)$ with values in E_m, such that $w(s) - w(t)$ for $s > t$ does not depend on the σ-algebra $\mathfrak{F}_t$ and

$$\xi(t) = Y^*(t)\int_0^T Q(t,u)\,dw(u), \tag{7}$$

where $Y^*(t)$ denotes the matrix inverse to the matrix $Y(t)$.

3) There exists an $\mathfrak{F}_t$-measurable random function $g(t)$ with values in E_m, for which, with probability 1,

$$\int_0^T \|g(t)\|^2\,dt < \infty; \tag{8}$$

$$f\left(t, Y^*(t)\int_0^t Y(s)\xi(s)\,ds\right) = Y^*(t)\int_0^T Q(t,u)g(u)\,du. \tag{9}$$

Theorem. Let the differential equations (1) and (2) be given in the m-dimensional Euclidean space E_m, where the Gaussian process $\xi(t)$, the function $f(t,x)$, and the linear operator $L(t)$ satisfy the conditions formulated above, and, in addition, let conditions 1)–3) be fulfilled. Then the measure μ_2 is absolutely continuous with respect to the measure μ_1 and

$$\frac{d\mu_2}{d\mu_1}[x_1] = \exp\left\{-\int_0^T (g(t), dw(t)) - \frac{1}{2}\int_0^T \|g(t)\|^2\,dt\right\}. \tag{10}$$

The results of this theorem can be applied to solutions of equations of order higher than the first.

Let $z_1(t)$ and $z_2(t)$ be random processes with values in E_m satisfying respectively the equations

$$d^n z_2(t)/dt^n + L_1(t)[z_2(t)] + F(t, z_2(t), z_2'(t), \dots, z_2^{(n-1)}(t)) = \eta(t) \tag{11}$$

$$(0 \leqslant t \leqslant T), \qquad z_2(0) = z_2'(0) = \dots = z_2^{(n-1)}(0) = \eta(0) = 0;$$

$$d^n z_1(t)/dt^n + L_1(t)[z_1(t)] = \eta(t) \qquad (0 \leqslant t \leqslant T), \tag{12}$$

$$z_1(0) = z_1'(0) = \dots = z_2^{(n-1)}(0) = \eta(0) = 0.$$

Here $L_1(t)$ is a linear differential operator of order $(n-1)$ with coefficients continuously depending on t and acting in E_m: if $y(t)$ is an m-dimensional function with values in E_m, differentiable $n-1$ times, then

$$L_1(t)[y(t)] = \sum_{k=0}^{n-1} C_k(t) y^{(k)}(t), \tag{13}$$

where $C_k(t)$, $k = 0, 1, \dots, n-1$, are matrices of order m. Further, the function $F(t, z_2(t), z_2'(t), \dots, z_2^{(n-1)}(t))$ is defined and continuous jointly in all variables in the domain $[0,T] \times E_m \times \dots \times E_m$, takes its values in E_m, and satisfies the condition

$$\sum_{j=0}^{n-1}\sum_{i=0}^{m}\left\|\frac{\partial F(t, z_2^{(0)}(t), z_2^{(1)}(t), \dots, z_2^{(n-1)}(t))}{\partial z_i^{(j)}}\right\| < \infty \tag{14}$$

($z_i^{(j)}$ is the i-th component of the vector $z^{(j)}$), and $\eta(t)$ is the Gaussian process with values in E_m considered above.

The case under consideration is reduced to the preceding one by introducing the processes $x_1(t)$ and $x_2(t)$ in the space $\mathbf{E}_{mn} = \mathbf{E}_m \times \cdots \times \mathbf{E}_m$, connected with the processes $z_1(t)$ and $z_2(t)$ in the following way:

$$x_i(t) = \left[z_i(t),\, dz_i(t)/dt, \dots, d^{(n-1)} z_i(t)/dt^{n-1}\right]. \tag{15}$$

The processes $x_i(t)$ $(i = 1, 2)$ already satisfy differential equations of the form (1) and (2) in $\mathbf{E}_{mn}$, where the function $f(t, x_2(t))$ is defined on $[0, T] \times \mathbf{E}_{mn}$, takes its values in $\mathbf{E}_{mn}$, and has the form

$$f(t, x_2(t)) = [0, \dots, 0, F(t, z_2(t), z_2'(t), \dots, z_2^{(n-1)}(t))],$$

the Gaussian process

$$\xi(t) = [0, \dots, 0, \eta(t)]$$

is defined on $[0, T]$ and takes its values in $\mathbf{E}_{mn}$, and the linear operator $L(t)$ acts in $\mathbf{E}_{mn}$ and has the form

$$L(t) = \begin{pmatrix} (0), & (I), & \dots, & (0) \\ (0), & (0), & \dots, & (0) \\ \cdot & \cdot & \cdot & \cdot \\ (0), & (0), & \dots, & (I) \\ C_0(t), & C_1(t), & \dots, & C_{n-1}(t) \end{pmatrix}.$$

Here each box of the form (0) is an m-dimensional matrix with zero entries, and a box of the form (I) is the m-dimensional identity matrix. If μ_{z_1} and μ_{z_2} denote the measures generated respectively by the processes $z_1(t)$ and $z_2(t)$, then, provided the conditions of the theorem are satisfied for equations (1) and (2) in $\mathbf{E}_{mn}$, the measure μ_{z_2} will be absolutely continuous with respect to the measure μ_{z_1}, and

$$\frac{d\mu_{z_2}}{d\mu_{z_1}}(z_1) = \frac{d\mu_2}{d\mu_1}(x_1),$$

while the density $\dfrac{d\mu_2}{d\mu_1}(x_1)$ will have the form (10), where the scalar product and the norm must be understood in the space $\mathbf{E}_{mn}$, and the random function $g(t)$ and the Wiener process $w(t)$ take their values in $\mathbf{E}_{mn}$ and are defined, respectively, by formulas (9) and (7).

It should be noted that the results of the present paper remain valid if one assumes that the space $\mathbf{E}_m$ is a separable Hilbert space $\mathbf{E}$, and equations (1), (2), (11), (12) are differential equations in the Hilbert space $\mathbf{E}$.

Donets Computing Center
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.