ILL-POSED PROBLEMS AND GEOMETRIES OF BANACH SPACES

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Abstract

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ILL-POSED PROBLEMS AND GEOMETRIES OF BANACH SPACES

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1. Statement of the problem. Let X be a reflexive space; Y a separable locally convex space, and A a linear continuous operator mapping X into Y, such that there exists an inverse operator A^{-1}, which, generally speaking, is not continuous. Consider the operator equation of the first kind

$$Ax = y_0, \qquad x \in X, \qquad y_0 \in R(A) \subset Y. \tag{1}$$

Suppose that y_0 is not known to us, and instead of it there is given a basis of a filter of neighborhoods $\{V_\delta\}$ of the point y_0, where each V_δ is a convex closed set: $V_\delta = \overline{\text{co}}(V_\delta)$.

It is required, from the given V_δ, to find an approximate solution x_δ of equation (1) such that $x_\delta \to x_0$, where x_0 is the exact solution of equation (1), i.e. $x_0 = A^{-1}y_0$.

2. The residual method. The idea of a stable solution of an operator equation of the first kind

$$\bar{A}[x] = u_0 \qquad (x \in X, \quad u_0 \in U), \tag{2}$$

where $\bar{A}[x]$ is a continuous operator from X into U, satisfying the uniqueness condition: $\bar{A}[x_1] \neq \bar{A}[x_2]$ if $x_1 \neq x_2$; X, U are linear normed spaces, was first put forward by A. N. Tikhonov ($^{1-3}$). Its basis was the additional assumption that the exact solution x_0: $\bar{A}[x_0] = u_0$ of equation (2) belongs to some class of well-posedness (stabilization) m, $m \subset X$.

In (^{3}), for example, for solving equation (2) the idea of a compact embedding is developed. This idea may be interpreted, as was done by V. K. Ivanov in (^{5}) for equation (2), as follows: it was assumed that there is a Hilbert space Z, which is mapped into X by means of a linear completely continuous operator B. It was further assumed that the exact solution x_0 of equation (2) belongs to the range $R(B)$ of the operator B in the space X: $x_0 \in R(B)$. The set $R(B)$,

$R(B) \subset X$, in this case plays the role of the stabilization class in the space X, and, generally speaking, $R(B) \neq X$. The residual method for solving equation (2) (see (^{5})) consisted in finding an element z_δ with minimal norm on the set $\Omega_\delta : \ \Omega_\delta = \{z : \ \|Cz - u_\delta\| \leq \delta\}$, where $C = \bar{A}B$ is a continuous operator acting from Z into U, $u_\delta \in U$, $\|u_0 - u_\delta\| \leq \delta$.

Let us note that if the operator $\bar{A}$, in addition, is linear, then for a stable solution of equation (2) it is sufficient to have merely a continuous embedding, i.e. the embedding operator B, mapping Z into X, may be assumed linear and continuous (see (^{13,14})). In connection with the idea of a continuous embedding, the case $Z = X$, and B the identity operator on Z, is of interest. In this case the stabilization class $R(B)$ is maximal in the space X, or, in other words, $R(B) = X$. This circumstance is very convenient, since the necessity of checking whether the exact solution x_0 of equation (2) belongs to the stabilization class $R(B)$ is eliminated.

In the work of V. A. Morozov, for example (see (^{13})), the space X is assumed to be Hilbert, and the stabilization class to coincide with the whole X,

and $\overline{A}$ is a linear continuous operator mapping X into a linear normed space U, and under these assumptions a stable solution of equation (2) is given by the method of A. N. Tikhonov. However, even the assumption that the spaces X are Hilbert spaces is inessential (see (^{14}, ^{16})). It is enough that X satisfy the Efimov-Stechkin condition (see (^{17})). An example of an Efimov-Stechkin space may be, for instance, the space L_p $(p > 1)$.

A natural question arises: in which spaces may one suppose the class of stabilization $R(B)$ to coincide with the whole space X? This note is devoted to the solution of this question. Therefore, in what follows we shall consider only the case $Z = X$ and B is the identity operator on X. Then the residual method, as applied to the solution of the problem posed in Sec. 1, consists in finding a point of minimum of the functional

$$K[x] = \|x\| \tag{3}$$

under the condition

$$Ax \in V_\delta. \tag{4}$$

Since X is reflexive, the point of minimum exists (see (^{10})). This point is unique if X is strictly convex and reflexive (see (^{10})). Therefore, in what follows we shall always assume X to be strictly convex and reflexive. The points of minimum will be called solutions of the problem posed in Sec. 1, obtained by the residual method, and will be denoted by x_δ. In what follows, the problem posed in Sec. 1 will be denoted by θ: $\theta = \theta[X, Y, x_0, A, \{V_\delta[Ax_0]\}]$, where $X \in \mathfrak{X}$, $\mathfrak{X}$ is the class of reflexive strictly convex spaces; $Y \in \mathfrak{Y}$, $\mathfrak{Y}$ is the class of separable locally convex spaces; $x_0 \in X$; $A \in \mathcal{L}(X, Y)$, $\mathcal{L}(X, Y)$ is the space of linear continuous one-to-one operators acting from X into Y; $\{V_\delta[Ax_0]\} \in \mathfrak{B}$, $\mathfrak{B}$ is the totality of all such bases of the filter of neighborhoods of the point $Ax_0 \in Y$, each of which consists only of convex closed sets of the space Y. By Θ_X we shall denote the set of problems θ for which X—a reflexive strictly convex space —is fixed, whereas the remaining components of the problem θ range over the corresponding domains of definition.

3. Basic definitions.

Definition 1. The residual method will be called **stable for the problem** θ if the solutions x_δ of this problem, obtained by the residual method, converge strongly to the exact solution x_0 of equation (1).

Remark. In this definition all components of the problem θ are regarded as fixed.

Definition 2. The residual method will be called **stable on the space** X if it is stable for every problem θ: $\theta \in \Theta_X$.

Definition 3. We shall say that the space X **satisfies condition (A1)** if from the fact that $\{x_n\} \subset X$, $x_0 \in X$,

$$x_n \xrightarrow{\text{w}} x_0 \quad \text{and} \quad \|x_n\| = \inf_{x \in \overline{\text{co}}(x_n, x_{n+1}, \dots)} \|x\|$$

it follows that $x_n \to x_0$.

Definition 4. We shall say that the space X **satisfies condition (A2)** if from the fact that $\{x_n\} \subset X$, $x_0 \in X$, $x_n \xrightarrow{\text{w}} x_0$, $\|x_n\| = 1$ and

$$\lim_{n\to\infty} \sup_{k \geq n} \rho(x_k, H_n) = 0,$$

where $H_n = \{x : f_n(x) = 1,\ f_n \in X^*,\ \|f_n\| = 1\}$ is a closed hyperplane supporting the ball $s = \{x : \|x\| \leq 1\}$ at the point x_n, it follows that $x_n \to x_0$.

Definition 5. We shall say that the space X **satisfies condition (A3)** if from the fact that $\{x_n\} \subset X$, $x_0 \in X$, $\|x_n\| = 1$,

$$x_n \xrightarrow{\text{w}} x_0 \quad \text{and} \quad \lim_{n\to\infty} \rho(x_0, H_n) = 0,$$

where H_n is a closed hyperplane supporting the ball $s = \{x : \|x\| \leq 1\}$ at the point x_n, it follows that $x_n \to x_0$.

Definition 6. We shall say that the space X satisfies condition (A4) if from the fact that $\{x_n\} \subset X$, $x_0 \in X$, $\|x_n\| = \|x_0\| = 1$, and $x_n \xrightarrow{\text{w}} x_0$, it follows that $x_n \to x_0$.

Definition 7. The space X is called an E-space (see (9)) if it is reflexively strictly convex and satisfies condition (A4). E-spaces are the largest known class of Banach spaces for which the operator P of metric projection onto a closed convex set is well-posed in the sense of Hadamard (see (6)).

4. Main theorems.

Theorem 1. Let X be a separable reflexive strictly convex space. In order that the residual method be stable on the space X, it is necessary that X satisfy condition (A1).

Theorem 2. Let X be a reflexive strictly convex space. In order that the residual method be stable on the space X, it is sufficient that X satisfy condition (A4).

Theorem 3. Let X be a reflexive strictly convex space; then conditions (A1), (A2), (A3), (A4) on the space X are equivalent.

Theorem 4. Let X be a separable reflexive strictly convex space. In order that the residual method be stable on the space X, it is necessary and sufficient that X be an E-space.

5. Finite-dimensional approximations of problem θ. Let X be a separable E-space and

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq X \tag{5}$$

be an increasing chain of finite-dimensional subspaces of the space X such that

$$\overline{\bigcup_{n=1}^{\infty} X_n} = X. \tag{6}$$

The existence of such an increasing chain follows easily from the separability of X. We shall solve the problem of finding a point of minimum of the functional

$$K[x] = \|x\| \tag{7}$$

under the conditions

$$x \in X_n, \qquad Ax \in V_\delta. \tag{8}$$

From the fact that X is an E-space, there follows the existence and uniqueness of the point of minimum. Denote this point by x_δ^n.

Theorem 5. The sequence of elements $\{x_\delta^n\}$ converges strongly to the approximate solution x_δ of problem θ:

$$x_\delta^n \to x_\delta \quad \text{as } n \to \infty. \tag{9}$$

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Note: Figure translations are in progress. See original paper for figures.