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Abstract

Full Text

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A HARTOGS THEOREM FOR GENERALIZED ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Let G be an open set in the n-dimensional complex space C^n. In the set G we consider the following system of equations with partial complex derivatives (see (3)):

$$\partial w / \partial z_j^* = f_j w \tag{1}$$

or

$$\partial w / \partial z_j = f_j w \tag{2}$$

$(j = 1, \dots, n)$. We assume that:

1°. The functions $\mathrm{Re}[f_j]$ and $\mathrm{Im}[f_j]$ can be (locally) represented in the form of power series.

2°. $\partial f_j / \partial z_k^* = \partial f_k / \partial z_j^*$, $\forall j$, $\forall k$, in the case of (1), or $\partial f_j / \partial z_k = \partial f_k / \partial z_j$, $\forall j$, $\forall k$, in the case of (2).

Concerning solutions $w = w(z_1, \dots, z_n)$ we shall assume:

1°. If $(z_1^0, \dots, z_n^0) \in G$, then let G_k, $k = 1, \dots, n$, be the intersection of the set G with the plane

$$\{(z_1, \dots, z_n);\ z_1 = z_1^0, \dots, z_{k-1} = z_{k-1}^0,\ z_{k+1} = z_{k+1}^0, \dots, z_n = z_n^0\}.$$

In the set G_k consider the function g, defined as follows:

$$g(z_k) = w(z_1^0, \dots, z_{k-1}^0, z_k, z_{k+1}^0, \dots, z_n^0).$$

We assume that g has partial complex derivatives with respect to z_k^* (or, in the case of (2), with respect to z_k) in the weak sense. This means (see (4)) that there exist complex numbers $\partial w(z_1^0, \dots, z_n^0)/\partial z_k^*$ or $\partial w(z_1^0, \dots, z_n^0)/\partial z_k$, from which the quantities $\int_{\partial B} g(z_k)\, dz_k/2i\, mB$ or

$$\int_{\partial B} g(z_k)\, dz_k^*/-2i\, mB$$

differ arbitrarily little, if $B(\subset G_k)$ is located in a sufficiently small neighborhood of the point z_k^0 (mB is the measure of the set B)*.

2°. At every point of the set G, the function w and its derivative in the weak sense (see 1°) satisfy the system (1) or (2).

A solution of the system (1) or (2) with these properties is called a weak solution.

Theorem 1. *If w is a weak solution of the system (1) or (2), then w is a continuous function in G. Moreover, w can be represented in the form of a power series (the variables of this series are x_j and y_j, $j = 1, \dots, n$). The derivative in the weak sense can be understood in the classical sense.*

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* It is sufficient to consider such domains B for which Int B is simply connected. Moreover, we consider only such B for which the Ostrogradsky–Gauss theorem is valid.

Proof. Let

$$\omega(x_1^0 + iy_1, \dots, x_n^0 + iy_n) = 0,$$

and let ω, in case (1), be a solution of the system

$$\partial\omega/\partial z_j^* = f_j \tag{3}$$

or, in case (2), a solution of the system

$$\partial\omega/\partial z_j = f_j. \tag{4}$$

(see ([3])). The functions Re[ω], Im[ω] can be represented in the form of power series. Therefore the partial complex derivatives $\partial\omega/\partial z_j^*$ and $\partial\omega/\partial z_j$ exist in the classical sense:

$$\partial\omega/\partial z_j^* = \frac{1}{2}(\partial\omega/\partial x_j + i\partial\omega/\partial y_j),$$

$$\partial\omega/\partial z_j = \frac{1}{2}(\partial\omega/\partial x_j - i\partial\omega/\partial y_j).$$

If the derivatives exist in the classical sense, then the derivatives in the weak sense also exist and are equal to the classical derivatives. If h_1, h_2 have derivatives in the weak sense, then $h_1^0 h_2$ also has derivatives in the weak sense; $h_1 h_2$ is differentiable in the weak sense if h_2 is differentiable in the weak sense and h_1 is holomorphic. In this case the usual rule of differentiation of composite functions is valid (see ([4])).

In particular,

$$\Phi = w \exp(-\omega)$$

is differentiable in the weak sense:

$$\partial\Phi/\partial z_j^* = \exp(-\omega)(\partial w/\partial z_j^* - w\,\partial\omega/\partial z_j^*) = \exp(-\omega)(f_{jw} - w f_j) = 0$$

or (in case (2))

$$\partial\Phi/\partial z_j^* = \exp(-\omega)(\partial w/\partial z_j^* - w\,\partial\omega/\partial z_j^*) = \exp(-\omega)(f_{jw} - w f_j) = 0.$$

If $\partial\Phi/\partial z_k^* = 0$ at every point of the set G_h, then Φ is a holomorphic function in G_h (see ([4])). If Φ satisfies all the equations $\partial\Phi/\partial z_k^* = 0$, $\forall k = 1, \dots, n$, then Φ, by Hartogs' theorem (see ([1])), is a continuous function in the set G, and it can be (locally) represented in the form of a power series (with respect to $x_j, y_j,\ j = 1, \dots, n$). With the aid of the functions $\tilde{z}_k = z_k^*$, one can obtain the derivative $\partial/\partial z_k$ from the derivative $\partial/\partial \tilde{z}_k^*$. Therefore we obtain: if $\partial\Phi/\partial z_k = 0$, $\forall k = 1, \dots, n$, then the function Φ is continuous and can be represented in the form of a power series (with respect to $x_j, y_j,\ j = 1, \dots, n$). From $w = \Phi \exp\omega$ it follows in both cases that Φ is a continuous function, and that Φ can be (locally) represented in the form of a power series. The theorem is proved.

If the functions f_j have compact supports, then there exist (see ([2])) solutions of equation (3) or (4) under assumption 2° and provided that the f_j possess continuous partial derivatives of first order. In this case the following holds.

Theorem 2. *A weak solution of system (1) or (2) is continuous and has continuous partial derivatives of first order. It follows from this, in particular, that derivatives in the weak sense may be understood in the classical sense.*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.