ON GRAVITATIONAL STRESSES

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Abstract

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I. G. FIKHTENGOL' TS

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Let us consider certain quantities associated with the curvature tensor and interpreted as gravitational stresses. In considering the question of gravitational stresses, harmonic coordinates are of fundamental importance.

Introduce the quantities

$$P_{\mu\nu} = R_{\mu\nu} - K_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the curvature tensor of rank two and

$$K_{\mu\nu} = \frac{1}{2}\left(g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x_\alpha \partial x_\beta} - g^{\alpha\beta}g^{\sigma\tau}\frac{\partial g_{\mu\sigma}}{\partial x_\alpha}\frac{\partial g_{\nu\tau}}{\partial x_\beta} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\Gamma^\alpha\right). \tag{2}$$

Here, as usual, $g_{\mu\nu}$ is the fundamental tensor,

$$\Gamma^\alpha = g^{\mu\nu}\Gamma^\alpha_{\mu\nu} \tag{3}$$

and $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols of the second kind. Greek indices take the values $0, 1, 2, 3$. Summation over identical Greek indices from 0 to 3 is assumed.

Recall (see ([1])) that the harmonic-coordinate conditions can be represented in the form

$$\Gamma^\alpha = 0. \tag{4}$$

From equalities (1) and (2) it follows that

$$P_{\mu\nu} = -\frac{1}{2}\left(g_{\mu\alpha}\frac{\partial \Gamma^\alpha}{\partial x_\nu} + g_{\nu\alpha}\frac{\partial \Gamma^\alpha}{\partial x_\mu}\right) + \frac{1}{4}g^{\alpha\beta}g^{\sigma\tau}\frac{\partial g_{\alpha\sigma}}{\partial x_\mu}\frac{\partial g_{\beta\tau}}{\partial x_\nu} - \frac{1}{2}g^{\alpha\beta}g^{\sigma\tau}\left(\frac{\partial g_{\mu\alpha}}{\partial x_\sigma}\frac{\partial g_{\beta\tau}}{\partial x_\nu} + \frac{\partial g_{\nu\alpha}}{\partial x_\sigma}\frac{\partial g_{\beta\tau}}{\partial x_\mu} - \frac{\partial g_{\mu\alpha}}{\partial x_\sigma}\frac{\partial g_{\nu\tau}}{\partial x_\beta}\right). \tag{5}$$

Thus, in harmonic coordinates (and only in harmonic coordinates), $P_{\mu\nu}$ contains no second derivatives of the components of the fundamental tensor $g_{\mu\nu}$.

There are very substantial grounds for interpreting the quantities $\frac{1}{\chi}P_{\mu\nu}$, where χ is Einstein' s gravitational constant, as gravitational stresses.

From equalities (5) it follows that, for any static field,

$$P_{c0} = 0, \qquad P_{0i} = 0. \tag{6}$$

Put

$$K = g^{\mu\nu}K_{\mu\nu}, \qquad P = g^{\mu\nu}P_{\mu\nu}. \tag{7}$$

Then, according to (2) and (7), we obtain

$$K = g^{\alpha\beta}\frac{\partial^2 \ln\sqrt{-g}}{\partial x_\alpha \partial x_\beta} - \frac{\partial \ln\sqrt{-g}}{\partial x_\alpha}\Gamma^\alpha, \tag{8}$$

i.e., K is the d' Alembert operator applied to $\ln\sqrt{-g}$, where g is the determinant composed of the components of the fundamental tensor $g_{\mu\nu}$.

From (5) and (7) it follows that

$$P = -\frac{\partial \Gamma^\alpha}{\partial x_\alpha} - g^{\alpha\beta}\Gamma^\sigma_{.\alpha\tau}\Gamma^\tau_{\beta\sigma}. \tag{9}$$

The expression for P becomes especially simple in harmonic coordinates.

For the Einstein tensor the equality holds

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = P_{\mu\nu} + Q_{\mu\nu}, \tag{10}$$

where

$$Q_{\mu\nu} = K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{11}$$

In these formulas the quantities $K_{\mu\nu}$ and $P_{\mu\nu}$ have, respectively, the forms (2) and (5), while R is the invariant of the curvature tensor, with

$$R = K + P. \tag{12}$$

Einstein' s equations of gravitation can be represented in the form

$$P_{\mu\nu} + Q_{\mu\nu} = -\chi T_{\mu\nu}, \tag{13}$$

where $T_{\mu\nu}$ is the mass tensor.

In our preceding work ([2]) we considered Friedmann-Lobachevsky space and showed that in this space one can introduce the concept of gravitational pressure. The indicated result is obtained here as well, in a more general formulation of the question.

As applied to Friedmann-Lobachevsky space one may put (see ([1,2]))

$$g_{\mu\nu} = \psi\eta_{\mu\nu}, \tag{14}$$

where $\psi = \psi(x_0, x_1, x_2, x_3) > 0$, $\eta_{00} = 1$, $\eta_{0i} = 0$, $\eta_{ik} = -\delta_{ik}$. Here, as usual, $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$. Latin indices take the values $1, 2, 3$.

For a space whose fundamental tensor has the form (14), the equalities hold

$$g^{\alpha\beta} g^{\sigma\tau} \frac{\partial g_{\mu\sigma}}{\partial x_\alpha} \frac{\partial g_{\nu\tau}}{\partial x_\beta} + \frac{\partial g_{\mu\nu}}{\partial x_\alpha} \Gamma^\alpha = 0, \tag{15}$$

and therefore (see (2) and (15))

$$K_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x_\alpha \partial x_\beta}. \tag{16}$$

Relations (15) and (16), which hold for the metric (14), cease to be valid when coordinates other than conformally Galilean ones are used.

From (14) and (16) it follows that in Friedmann-Lobachevsky space

$$K_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \left(\frac{\partial^2 \ln\psi}{\partial x_\alpha \partial x_\beta} + \frac{\partial \ln\psi}{\partial x_\alpha} \frac{\partial \ln\psi}{\partial x_\beta} \right) g_{\mu\nu}. \tag{17}$$

According to (5) and (14) we obtain

$$P_{\mu\nu} = \frac{\partial^2 \ln\psi}{\partial x_\mu \partial x_\nu} - \frac{1}{2} \frac{\partial \ln\psi}{\partial x_\mu} \frac{\partial \ln\psi}{\partial x_\nu}. \tag{18}$$

From equalities (7), (12), (17), and (18) it follows that for the space under consideration

$$R = 3g^{\alpha\beta} \left(\frac{\partial^2 \ln\psi}{\partial x_\alpha \partial x_\beta} + \frac{1}{2} \frac{\partial \ln\psi}{\partial x_\alpha} \frac{\partial \ln\psi}{\partial x_\beta} \right) \tag{19}$$

and (see (11), (17), and (19))

$$Q_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{20}$$

$$\Lambda = g^{\alpha\beta}\left(\frac{\partial^2 \ln\psi}{\partial x_\alpha \partial x_\beta} + \frac{1}{4}\frac{\partial \ln\psi}{\partial x_\alpha}\frac{\partial \ln\psi}{\partial x_\beta}\right). \tag{21}$$

Taking into account (10), (18), (20), and (21), we arrive at the relations considered in detail in work ([2]).

For the solution of Einstein' s gravitational equations obtained in the book ([1]), the equality (see ([2])) is valid:

$$\Lambda = -\frac{\chi}{c^2}p, \tag{22}$$

where

$$p = \frac{c^2\rho}{3}\sqrt[4]{\psi}. \tag{23}$$

Here p is the gravitational pressure and ρ is the invariant mass density.

Let us now find the quantities $P_{\mu\nu}$ for the Schwarzschild space. For a spherically symmetric gravitational field (in the region outside the mass), in harmonic coordinates the following equalities hold:

$$g_{00} = c^2\frac{r-\alpha}{r+\alpha}, \qquad g_{0i} = 0, \qquad g_{ik} = -\left(1+\frac{\alpha}{r}\right)^2\delta_{ik} - \frac{r+\alpha}{r-\alpha}\frac{\alpha^2}{r^4}x_i x_k. \tag{24}$$

Here $\alpha = \gamma m/c^2$ is the gravitational radius of the mass m, which is the source of the field (γ is Newton' s gravitational constant), and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

For the Schwarzschild space the equalities (6) hold and, according to (5) and (24),

$$P_{ik} = -\frac{\alpha^4}{2r^4(r^2-\alpha^2)}\delta_{ik} + \frac{\alpha^2(4r^4 - \alpha^2 r^2 - \alpha^4)}{2r^6(r^2-\alpha^2)^2}x_i x_k. \tag{25}$$

For any static field in the region outside the masses,

$$Q_{00} = 0, \qquad Q_{0i} = 0; \tag{26}$$

$$Q_{ik} = -P_{ik}. \tag{27}$$

It follows from (25) and (27) that, for the Schwarzschild space,

$$Q = 2\alpha^2/r^2(r+\alpha)^2, \tag{28}$$

where, of course,

$$Q = g^{\mu\nu} Q_{\mu\nu}. \tag{29}$$

Let us turn to the consideration of the quantities $P_{\mu\nu}$ as applied to the astronomical problem of an isolated system of masses.

We shall use the approximate solution of Einstein' s gravitational equations obtained in the book ([1]). In the first approximation one may put

$$g_{00} = c^2(1 - 2U/c^2), \qquad g_{0i} = 4U_i/c^2, \qquad g_{ik} = -(1 + 2U/c^2)\delta_{ik}. \tag{30}$$

In the formulas given, U is the Newtonian potential and U_i is the vector potential of the gravitational field.

Proceeding from (5) and (30), in the first approximation we obtain

$$\begin{aligned} P_{00} &= -\frac{2}{c^4}\left(3\frac{\partial U}{\partial t}\frac{\partial U}{\partial t} - 4\frac{\partial U_i}{\partial x_k}\frac{\partial U_k}{\partial x_i}\right), \\ P_{0i} &= -\frac{2}{c^4}\left(3\frac{\partial U}{\partial t}\frac{\partial U}{\partial x_i} + 4\frac{\partial U_k}{\partial x_i}\frac{\partial U}{\partial x_k}\right), \\ P_{ik} &= -\frac{2}{c^4}\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_k}. \end{aligned} \tag{31}$$

Summation from 1 to 3 is assumed over identical Latin indices.

Formulas (31) represent the well-known analogy with formula (56.55) of the book ([1]).

For the quantities $Q_{\mu\nu}$ we have, according to Einstein' s gravitational equations,

$$Q_{\mu\nu} = -\chi T_{\mu\nu} - P_{\mu\nu}. \tag{32}$$

The foregoing gives grounds for interpreting $\frac{1}{\varkappa} P_{\mu\nu}$ as quantities characterizing gravitational stresses.

The decomposition of the Einstein tensor into the sum $P_{\mu\nu}$ and $Q_{\mu\nu}$ is not generally covariant. The quantities $P_{\mu\nu}$ (and, correspondingly, $Q_{\mu\nu}$) form a tensor with respect to Lorentz transformations. Moreover, what appears to be quite essential is that in a harmonic system of coordinates (and only in a harmonic one) $P_{\mu\nu}$ (see (5)) contains no second derivatives of the components of the fundamental tensor $g_{\mu\nu}$.

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Leningrad Institute
of Precision Mechanics and Optics

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REFERENCES

[1] V. A. Fock, *The Theory of Space, Time and Gravitation*, Moscow, 1961. [2] I. G. Fichtenholtz, DAN, **191**, No. 5 (1970).

Note: Figure translations are in progress. See original paper for figures.