

# ON THE STABILITY OF PERIODIC MOTIONS IN THE TRANSCENDENTAL CASE OF TWO PURELY IMAGINARY CHARACTERISTIC EXPONENTS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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**ON THE STABILITY OF PERIODIC MOTIONS IN THE TRANSCENDENTAL CASE OF TWO PURELY IMAGINARY CHARACTERISTIC EXPONENTS**

*(Presented by Academician V. I. Smirnov on 9 VI 1969)*

1. We consider a system of differential equations

$$d\xi/dt = E(\xi, t), \tag{1}$$

where  $E(\xi, t)$  is an  $(n + 2)$ -dimensional vector-function of a vector argument  $\xi$  of the same dimension and of a scalar argument  $t$ , analytic in  $\xi$  in some neighborhood of the point  $\xi = 0$ , with  $E(0, t) \equiv 0$ , continuous and  $2\pi$ -periodic in  $t$ . We consider the question of Lyapunov stability of the solution  $\xi = 0$  of system (1) in the real domain. It is assumed that  $n$  characteristic exponents of the system of the linear approximation have negative real parts, while the remaining 2 are purely imaginary, equal to  $\pm\lambda i$ , where  $\lambda$  is irrational.

Under the assumptions made, system (1), by means of a nonsingular linear transformation with periodic coefficients, can be reduced to the form

$$\begin{aligned} \dot{x} &= i\lambda x + X(x, y, z_j, t), \\ \dot{y} &= -i\lambda y + \bar{X}(x, y, z_j, t), \\ \dot{z}_\nu &= \chi_\nu z_\nu + \gamma_\nu z_{\nu-1} + Z_\nu(x, y, z_j, t), \quad (\nu, j = 1, \dots, n), \end{aligned} \tag{2}$$

where  $X, Z_\nu$  are series in powers of  $x, y, z_j$  with continuous periodic coefficients, convergent in some neighborhood of the origin of coordinates, not containing terms below the 2nd dimension; moreover, to real solutions of system (1) there correspond complex conjugate variables  $x$  and  $y$ , i.e.  $x = \bar{y}$ , and  $\gamma_\nu$  are equal either to 0 or to 1. In system (2) and in what follows,  $\bar{X}$  denotes the function conjugate to  $X$ .

The problem of stability was considered by A. M. Lyapunov <sup>(1)</sup>. His results can be stated in the following way <sup>(2)</sup>.

For any natural  $N$ , system (2), by means of the transformation

$$u = x + \varphi(x, y, t), \quad v = y + \bar{\varphi}(x, y, t), \quad w_\nu = z_0 + \psi_0(x, y, t) \quad (\nu = 1, \dots, n), \quad (3)$$

where  $\varphi, \psi_\nu$  are polynomials in  $x, y$  with continuous periodic coefficients, of degree not exceeding  $2N$ , not containing terms below the 2nd dimension, can be brought to the form

$$\begin{aligned} \dot{u} &= i\lambda u + uP_{N-1}(uv) + \Phi(u, v, w_j, t), \\ \dot{v} &= -i\lambda v + v\bar{P}_{N-1}(uv) + \bar{\Phi}(u, v, w_j, t), \\ \dot{w}_\nu &= \chi_\nu w_\nu + \gamma_\nu w_{\nu-1} + \Psi_\nu(u, v, w_j, t) \quad (\nu, j = 1, \dots, n), \end{aligned} \quad (4)$$

where  $P_{N-1}$  is a polynomial of degree not exceeding  $N - 1$  in the product  $uv$ , without a constant term, with constant coefficients;  $\Phi, \Psi_\nu$  are series in powers of  $u, v, w_j$  with periodic coefficients, and the expansions of the functions  $\Phi(u, v, 0, t)$ ,  $\Psi_\nu(u, v, 0, t)$  contain no terms below the  $(2N + 1)$ -st dimension. If among the coefficients of the polynomials  $P_{N-1}$  there occur coefficients different from

purely imaginary, then the question of stability is decided by the sign of the real part of the first among such coefficients. A. M. Lyapunov called such cases algebraic.

We shall consider the case when, for any  $N$ , the coefficients  $P_{N-1}$  turn out to be purely imaginary, i.e.

$$P_{N-1}(z) = iH_{N-1}(z), \quad (5)$$

where  $H_{N-1}$  is a polynomial with real coefficients, and we shall assume that the coefficient of  $z$  is different from 0. In accordance with A. M. Lyapunov's terminology, we shall call the case under consideration generally transcendental.

Up to the present time, stability of the equilibrium position in the generally transcendental case has been proved for  $n = 0$  for canonical systems (3), and in general for systems satisfying a certain special condition <sup>(4)</sup>, which, as will be shown below, exhausts the case  $n = 0$ .

The transformation reducing system (2) to the form (4) is not unique. Namely, the coefficients of the polynomial  $\varphi$  at powers of the form  $x(xy)^k$  contain an arbitrary additive constant. One may also assume that  $\varphi$  and  $\psi_\nu$  depend on  $z_j$ . However, it is not difficult to show that the validity of (5), i.e. the fact that the situation is transcendental or algebraic, does not depend on the functions  $\varphi, \psi_\nu$  in (3).

2. Let  $\varepsilon > 0$  be sufficiently small. We shall consider system (4) in the domain  $|u| < \varepsilon$ ,  $|v| < \varepsilon$ ,  $|w_j| < \varepsilon$ . Choose the real constant  $\mu$  in such a way that the inequality

$$|k\mu - p| > \varepsilon^3 |k|^{-2}, \quad k, p \text{ arbitrary integers, } (k, p) \neq (0, 0). \quad (6)$$

is satisfied. Such  $\mu$  exist in the  $\varepsilon^2$ -neighborhood of the point  $\lambda$  for any  $\varepsilon$  (see (5)).

We reduce system (2) to the form (4). By the usual methods of the theory of formal expansions one can show that, if it is allowed that the function  $\varphi$  in (3) depends on  $z_j$ , then  $\varphi$  and  $\psi_\nu$  can be determined so that the expansion  $\Phi(u, v, w_j, t)$  contains no terms below dimension  $(2N + 1)$ . To system (4) we apply the method of successive approximations of Newton type (6). Fix a sufficiently large number  $N$  and introduce into consideration the following sequence of natural numbers  $N_s$  ( $s = 1, 2, \dots$ ):  $N_1 = N$ ,  $N_{s+1} = 2N_s$ .

We describe the  $s$ -th step. Suppose that, as a result of the first  $s - 1$  steps, we have arrived at the system

$$\begin{aligned} \dot{u}_s &= i\mu u_s + iu_s[(\lambda - \mu) + H_s(r_s)] + \Phi_s(u_s, v_s, w_{js}, t), \\ \dot{v}_s &= -i\mu v_s - iv_s[(\lambda - \mu) + H_s(r_s)] + \bar{\Phi}_s(u_s, v_s, w_{js}, t), \\ \dot{w}_{\nu s} &= \kappa_\nu w_{\nu s} + \Upsilon_\nu w_{\nu-1s} + \Psi_{\nu s}(u_s, v_s, w_{js}, t) \quad (\nu, j = 1, \dots, n), \end{aligned} \quad (7)$$

where

$$H_s(r_s) = \sum_{\sigma=1}^{N_s-1} a_s^{(\sigma)}(r_s) r_s^\sigma, \quad \Phi_s = \sum_{m=2N_s+1}^{\infty} M_s^{(k,l,m_j)}(r_s, t) u_s^k v_s^l w_j^{m_j},$$

$$\Psi_{\nu s} = \sum_{m=2}^{\infty} L_{\nu s}^{(k,l,m_j)}(r_s, t) u_s^k v_s^l w_j^{m_j}; \quad r_s = u_s v_s; \quad m = k + l + m_1 + \dots + m_n;$$

$$k, l, m_j \geq 0.$$

Here  $a_s^{(\sigma)}$ ,  $M_s^{(k,l,m_j)}$ , and  $L_{\nu s}^{(k,l,m_j)}$  are functions of  $r_s$ , analytic in the domain  $\mathfrak{M}_s$ :

$$|(\lambda - \mu) + H_s(r_s)| < \delta_s, \quad |w_{js}| < \Delta_s, \quad (8)$$

where  $\delta_s, \Delta_s$  are sufficiently small, and which can be expanded in formal series in powers of  $r_s$ . Here  $M_s^{(k,l,m_j)}$ ,  $L_{\nu s}^{(k,l,m_j)}$  are periodic in  $t$ , while  $a_s^{(\sigma)}$  do not depend on  $t$  and are real. Finally, the expansions of the functions  $\Psi_{\nu s}(u_s, v_s, 0, t)$  contain

no terms below dimension  $(2N_s + 1)$ . Obviously, for  $s = 1$  these assumptions are satisfied if system (4) is taken as the first approximation.

the  $s$ -th step consists in reducing system (7), by means of the transformation

$$\begin{aligned} u_{s+1} &= u_s + \varphi_s(u_s, v_s, w_{js}, t), & v_{s+1} &= v_s + \bar{\varphi}_s(u_s, v_s, w_{js}, t), \\ w_{\nu s+1} &= w_{\nu s} + \psi_{\nu s}(u_s, v_s, t) & (\nu, j &= 1, \dots, n), \end{aligned}$$

where

$$\varphi_s = \sum_{m=2N_s+1}^{4N_s} A_s^{(k,l,m_j)} u_s^k v_s^l w_{js}^{m_j}, \quad \psi_{\nu s} = \sum_{m=2N_s+1}^{4N_s} B_{\nu s}^{(k,l)} u_s^k v_s^l,$$

$A_s^{(k,l,m_j)}$ ,  $B_{\nu s}^{(k,l)}$  are functions, analytic in  $r_s$  in the domain  $\mathfrak{M}_s$  and periodic in  $t$ , to the system

$$\begin{aligned} \dot{u}_{s+1} &= i\mu u_{s+1} + iu_{s+1}[(\lambda - \mu) + H_s(r_{s+1})] + u_{s+1}G_s(r_{s+1}) + \Phi_{s+1}, \\ \dot{v}_{s+1} &= -i\mu v_{s+1} - iv_{s+1}[(\lambda - \mu) + H_s(r_{s+1})] + v_{s+1}\bar{G}_s(r_{s+1}) + \bar{\Phi}_{s+1}, \\ \dot{w}_{\nu s+1} &= \chi_\nu w_{\nu s+1} + \gamma_\nu u_{\nu-1 s+1} + \Psi_{\nu s+1}, \quad (\nu = 1, \dots, n). \end{aligned} \quad (9)$$

Here

$$G_s = i \sum_{\sigma=N_s}^{2N_s-1} a_{s+1}^{(\sigma)} r_{s+1}^\sigma, \quad a_{s+1}^{(\sigma)} = \frac{1}{2\pi i} \int_0^{2\pi} M_s^{(\sigma+1, \sigma, 0)} dt,$$

and  $\Phi_{s+1}$ ,  $\Psi_{\nu s+1}$  are functions of the same character as  $\Phi_s$ ,  $\Psi_{\nu s}$ , where  $\Phi_{s+1}(u_{s+1}, v_{s+1}, w_{js+1}, t)$ ,  $\Psi_{\nu s+1}(u_{s+1}, v_{s+1}, 0, t)$  do not contain in their expansions terms of dimension lower than  $(2N_{s+1} + 1)$ .

The functions  $\varphi$ ,  $\psi_{\nu s}$  are determined as the solution of the system

$$\begin{aligned} & \frac{\partial \varphi_s}{\partial t} + i \left( \frac{\partial \varphi_s}{\partial u_s} u_s - \frac{\partial \varphi_s}{\partial v_s} v_s - \varphi_s \right) (\lambda + H_s) + \sum_{j=1}^n \frac{\partial \varphi_s}{\partial w_{js}} \chi_j w_{js} = \\ & = \left[ u_{sG} s - \Phi_s + iu_s \left( \frac{\partial H_s}{\partial u_s} \varphi_s + \frac{\partial H_s}{\partial v_s} \bar{\varphi}_s \right) - \sum_{j=1}^n \frac{\partial \varphi_s}{\partial w_{js}} (\gamma_j w_{j+s} + \Psi_{js}) \right]_{2N_s+1}^{4N_s}, \end{aligned} \quad (10)$$

$$\begin{aligned} & \frac{\partial \psi_{\nu s}}{\partial t} + i \left( \frac{\partial \psi_{\nu s}}{\partial u_s} - \frac{\partial \psi_{\nu s}}{\partial v_s} v_s \right) (\lambda + H_s) - \chi_\nu \psi_{\nu s} = \\ & = \gamma_\nu \psi_{\nu-1 s} - [\Psi_{\nu s}(u_s, v_s, -\psi_{j s}, t)]_{2N_s+1}^{4N_s} \quad (\nu, j = 1, \dots, n), \end{aligned}$$

where the symbol  $[f]_p^q$  denotes those terms of the expansion of the function  $f$  whose dimension  $m$  satisfies the inequality  $p \leq m \leq q$ .

Equating in (10) the coefficients of the same powers of  $u_s, v_s, w_{j s}$ , to determine  $A_s^{(k,l,m_j)}, B_{\nu s}^{(k,l)}$  we obtain linear differential equations of the first order, from which the coefficients are determined as functions periodic in  $t$  by the formulas

$$A_s^{(k,l,0)} = (e^{2\pi i \omega_s} - 1)^{-1} \int_t^{t+2\pi} e^{i\omega_s(\tau-t)} \left( -M_s^{(k,l,0)} + C_s^{(k,l,0)} \right) d\tau \quad (k \neq l + 1), \quad (11)$$

$$A_s^{(k,l,m_j)} = \int_{+\infty}^t e^{(i\omega_s + d_s)(\tau-t)} \left( -M_s^{(k,l,m_j)} + C_s^{(k,l,m_j)} \right) d\tau \quad (m_1 + \dots + m_n > 0), \quad (12)$$

$$B_{\nu s}^{(k,l)} = \int_{-\infty}^t e^{(i\omega_s - \chi_\nu)(\tau-t)} \left( -L_{\nu s}^{(k,l,0)} + D_{\nu s}^{(k,l)} \right) d\tau \quad (\nu = 1, \dots, n), \quad (13)$$

where

$$\omega_s = (k - l - 1)\mu + (k - l - 1)[(\lambda - \mu) + H_s],$$

$$d_s = m_1 \chi_1 + \dots + m_n \chi_n \quad (2N_s + 1 \leq m \leq 4N_s), \quad (14)$$

whereas  $A_s^{(k,l,m_j)}, B_{\nu s}^{(k,l)}$  can be ordered in such a way that  $C_s^{(k,l,m_j)}, D_{\nu s}^{(k,l)}$  will be expressed in terms of coefficients preceding the one being determined. If, however, in (11)  $k = l + 1$ , then the corresponding coefficient is determined as any antiderivative of a certain periodic function with zero mean value.

It follows from (6) and (14) that, if  $\delta_s$  in (8) is sufficiently small, then  $A_s^{(k,l,m_j)}, B_{\nu s}^{(k,l)}$  are functions of  $r_s$ , analytic in the domain  $\mathfrak{M}_s$ , which can be expanded in formal series in powers of  $r_s$ .

As a result, we arrive at a system (9) satisfying all the assumptions made with respect to system (7), if  $G_s$  can be represented in the form  $G_s = iQ_s$ , where  $Q_s$

is a real analytic function. Let us show the possibility of such a representation. To this end we represent  $G_s$  in the form  $G_s = iQ_s + Q'_s$ , where  $Q_s, Q'_s$  are real analytic functions. Then system (9) takes the form

$$\begin{aligned} \dot{u}_{s+1} &= i\lambda u_{s+1} + iu_{s+1}H_{s+1}(r_{s+1}) + u_{s+1}Q'_{s+1} + \Phi_{s+1}, \\ \dot{v}_{s+1} &= -i\lambda v_{s+1} - iv_{s+1}H_{s+1}(r_{s+1}) + v_{s+1}Q'_s + \bar{\Phi}_{s+1}, \\ \dot{w}_{\nu s+1} &= \chi_\nu w_{\nu s+1} + \gamma_\nu w_{\nu-1 s+1} + \Psi_{\nu s+1} \quad (\nu = 1, \dots, n), \end{aligned} \quad (9')$$

where  $H_{s+1} = H_s + Q_s$ . Let

$$Q'_s = \sum_{\sigma=N_s}^{2N_s-1} q_s^{(\sigma)}(r_{s+1})r_{s+1}^\sigma,$$

where  $q_s^{(\sigma)}$  are formal series in powers of  $r_{s+1}$ . If, in the formal expansion of  $Q'_s$  thus obtained, there are no terms below dimension  $2N_s$  in  $r_{s+1}$ , then the function  $u_{s+1}Q'_s$  can be included in  $\Phi_{s+1}$ , and in system (9') it may be regarded as absent. If, however, the least order in the formal expansion of  $Q'_s$  is less than  $2N_s$ , then we arrive at a contradiction with the remark made at the end of § 1.

The sequences  $\delta_s, \Delta_s$  can be chosen so that the domains  $\mathfrak{M}_s$  are nested in one another, with  $\delta_s \rightarrow 0, \Delta_s \rightarrow \frac{1}{2}\varepsilon$ . To prove convergence of the successive approximations, the coefficients  $A_s^{(k,l,m_j)}, B_{\nu s}^{(k,l)}$  are estimated by formulas (11), (12), and (13). Small denominators appear in the coefficients  $A_s^{(k,l,0)}$ . These coefficients are estimated in the same way as in (7). In the remaining coefficients there are no small denominators, which makes it possible to obtain analogous estimates.

Letting  $s$  tend to infinity, we obtain the system

$$\dot{u}_\infty = i\mu u_\infty, \quad \dot{v}_\infty = -i\mu v_\infty,$$

$$\dot{w}_{\nu\infty} = \chi_\nu w_{\nu\infty} + \gamma_\nu w_{\nu-1\infty} + \Psi_{\nu\infty} \quad (\nu = 1, \dots, n), \quad (15)$$

defined on the invariant surface  $\mathfrak{M}_\infty$  for system (15), and moreover  $\Psi_{\nu\infty}(u_\infty, v_\infty, 0, t) \equiv 0$ . Since  $\text{Re } \chi_\nu < 0$ , all trajectories of system (15) on the surfaces  $\mathfrak{M}_\infty$  tend, as  $t \rightarrow +\infty$ , to the curves  $r_\infty = u_\infty v_\infty = r_\mu$ , with  $r_\mu \rightarrow 0$  as  $\mu \rightarrow \lambda$ . Thus we obtain the theorem:

**Theorem.** *The zero solution of system (1), in the general transcendental case of two purely imaginary characteristic exponents, is Lyapunov stable.*

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