THE MULTIDIMENSIONAL PLATEAU PROBLEM AND SINGULAR POINTS OF MINIMAL COMPACTS

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THE MULTIDIMENSIONAL PLATEAU PROBLEM AND SINGULAR POINTS OF MINIMAL COMPACTS

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1. Let $(\mathfrak{M}^n, A)$ be a compact pair, where $\mathfrak{M}^n$ is a compact, closed Riemannian manifold. In note ([4]) the classes $O^k(A, \mathfrak{L}, \mathfrak{L}')$, $R^k(A)$, $N^k(A)$ were introduced and the corresponding existence theorems for minimal compacts were formulated. It was also noted there that a special case of the class $O^k(A, \mathfrak{L}, \mathfrak{L}')$ is the class

$$O^k(A, \mathfrak{L}, 0) = \mathfrak{G}^k(A, \mathfrak{L}),$$

studied by Reifenberg ([1]) and Morrey ([2]). In the case when the coefficient group $\mathfrak{G} = U = \mathbf{R}^1(\mathrm{mod} 1)$, the class $N^k(A)$ coincides with the class $\mathfrak{G}^*$ considered by Reifenberg in ([1]).

In the present note a new class $P^k(A, \mathfrak{L}, \mathfrak{L}')$ is introduced, in which the multidimensional Plateau problem is posed and solved. In addition, the note indicates a connection between the homological structure of minimal compacts in the classes $O^k(A, \mathfrak{L}, \mathfrak{L}')$ and properties of the set Z of their singular points.

2. By $H_*(X; \mathfrak{G})$ we shall denote the Čech–Aleksandrov homology of the space X with coefficients in the Abelian group $\mathfrak{G}$. In what follows, by $\mathfrak{G}_C$ we denote the category of compact Abelian groups, and by $\mathfrak{G}_F$ the category of linear vector spaces over some field F. Consider a compact closed Riemannian manifold $\mathfrak{M}^n$, and let

$$A \subset \mathfrak{M}^n$$

be an arbitrary but fixed compact set. Let

$$\mathfrak{L} \subset H_{k-1}(A; \mathfrak{G})$$

be an arbitrary subgroup, $\mathfrak{G} \in \mathfrak{G}_C$ or $\mathfrak{G} \in \mathfrak{G}_F$. Consider the class $O^k(A, \mathfrak{L}, 0)$ (for the definition see ([4])) and suppose that it is nonempty if the subgroup $\mathfrak{L}$ is nontrivial. Let $\mathfrak{L}'$ be an arbitrary subgroup in the group of relative homologies $H_k(\mathfrak{M}^n, A; \mathfrak{G})$, and suppose that at least one of the subgroups $\mathfrak{L}$ and $\mathfrak{L}'$ is nontrivial. If $\mathfrak{L} = 0$, then the class $O^k(A, 0, 0)$, obviously, coincides with the class of all compacts such that $X \subset \mathfrak{M}^n$ and $A \subset X$.

Definition 1. Consider the triple $(A, \mathfrak{L}, \mathfrak{L}')$. We shall say that a compact $X \subset \mathfrak{M}^n$ belongs to the class $P^k(A, \mathfrak{L}, \mathfrak{L}')$ if

$$X \in O^k(A, \mathfrak{L}, 0)$$

and

$$\omega_*(H_k(X, A; \mathfrak{G})) \supset \mathfrak{L}',$$

where

$$\omega_* : H_k(X, A; \mathfrak{G}) \to H_k(\mathfrak{M}^n, A; \mathfrak{G})$$

is the homomorphism induced by the inclusion

$$\omega : (X, A) \to (\mathfrak{M}^n, A).$$

For $\mathfrak{L} = 0$ we obtain compact pairs (X, A) realizing relative cycles in the group $H_k(\mathfrak{M}^n, A; \mathfrak{G})$. It is clear that

$$P^k(\emptyset, 0, \mathfrak{L}') = O^k(\emptyset, 0, \mathfrak{L}'); \qquad P^k(A, \mathfrak{L}, 0) = O^k(A, \mathfrak{L}, 0).$$

If one puts $\mathfrak{L} = 0$ and $\mathfrak{L}' = \{\sigma\}$, a subgroup in $H_k(\mathfrak{M}^n, A; \mathfrak{G})$ generated by the element σ, then one obtains the class $P^k(A, 0, \{\sigma\})$, studied by Almgren in ([3]) for the case of k-spanning compacts and finitely generated coefficient groups. Let us note that the relation

$$\mathfrak{L}' \subset \operatorname{Im} \omega_*$$

is equivalent to the condition that

$$\mathfrak{L}' \subset \operatorname{Ker} \varphi_*,$$

where

$$\varphi_* : H_k(\mathfrak{M}^n, A; \mathfrak{G}) \to H_k(\mathfrak{M}^n, X; \mathfrak{G}).$$

The technique used in the proof of the existence theorems in the classes $O^k(A, \mathfrak{L}, \mathfrak{L}')$, $N^k(A)$, $R^k(A)$ (see ([4])) makes it possible to obtain an existence theorem also in the class $P^k(A, \mathfrak{L}, \mathfrak{L}')$.

3. **Theorem 1.** Let $\mathfrak{M}^n$ be a compact, closed Riemannian manifold of class C^p, where $p \geq 4$; let $A \subset \mathfrak{M}^n$ be an arbitrary compact set; let $\mathfrak{G} \in \mathfrak{G}_C$ or $\mathfrak{G}_F$; let k be an integer and $k \geq 3$; let $\mathfrak{L}$ and $\mathfrak{L}'$ be subgroups in the groups $H_{k-1}(A; \mathfrak{G})$ and $H_k(\mathfrak{M}^n, A; \mathfrak{G})$, respectively, where at least one of them is nontrivial.

Suppose that $P^k(A, \mathfrak{L}, \mathfrak{L}') \neq \emptyset$ and $\mu(A, \mathfrak{L}, \mathfrak{L}') < \infty$, where

$$\mu(A, \mathfrak{L}, \mathfrak{L}') = \inf \Lambda^k(X \setminus A), \quad X \in P^k(A, \mathfrak{L}, \mathfrak{L}').$$

Then there exists a compact set $X_0 \in P^k(A, \mathfrak{L}, \mathfrak{L}')$ such that $\Lambda^k(X_0 \setminus A) = \mu(A, \mathfrak{L}, \mathfrak{L}')$, and every point $x \in (X_0 \setminus A) \setminus Z$, where $\Lambda^k(Z) = 0$, possesses in X_0 a neighborhood homeomorphic to the k-dimensional disk D^k. Moreover, if $\mathfrak{M}^n \in C^4$, then these k-disks may be assumed to belong to the class C^3_μ for any $0 < \mu < 1$; if $\mathfrak{M}^n \in C^p_\mu$ for some $p \geq 4$ and some μ, $0 < \mu < 1$, then the k-disks may be assumed to belong to the class C^p_μ; finally, if $\mathfrak{M}^n \in C^\infty$ or is analytic, then the k-disks may be assumed to belong to the class C^∞ or to be analytic, respectively. In addition, $(X_0 \setminus A) \setminus Z$ is a submanifold in $\mathfrak{M}^n$ of the corresponding smoothness.

4. We proceed to the study of singular points of minimal compact sets $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$. Consider in the manifold $\mathfrak{M}^n$ the open submanifold $(X_0 \setminus A) \setminus Z$ (see the main theorem in ([4])), which, generally speaking, is not connected. Let

$$(X_0 \setminus A) \setminus Z = \bigcup_i \Pi_i,$$

where Π_i denotes the components of linear connectedness, which are differentiable submanifolds in $\mathfrak{M}^n$. If the submanifold $(X_0 \setminus A) \setminus Z$ consists of a finite number of linearly connected components, we shall denote their number by N.

Definition 2. A component Π_i is called $\mathfrak{G}$-orientable if

$$H_k(\overline{\Pi}_i, \partial\Pi_i; \mathfrak{G}) \neq 0,$$

where the closure $\overline{\Pi}_i$ is taken in $\mathfrak{M}^n$, $\partial\Pi_i = \overline{\Pi}_i \setminus \Pi_i$, $k = \dim X_0$. Let us note that here the group $\mathfrak{G}$ is precisely the group which characterizes the class $O^k(A, \mathfrak{L}, \mathfrak{L}')$.

Since

$$H_k(\overline{\Pi}_i, \partial\Pi_i; \mathfrak{G}) \cong H_k(\overline{\Pi}_i / \partial\Pi_i; \mathfrak{G}),$$

we are in fact speaking of the $\mathfrak{G}$-orientability of the compactification of the manifold Π_i by means of the point ω. The number of $\mathfrak{G}$-orientable components of the manifold $(X_0 \setminus A) \setminus Z$ will be denoted by N_0.

Theorem 2. Let $\mathfrak{M}^n$ be a compact, closed Riemannian manifold, $\mathfrak{M}^n \in C^4$, $\mathfrak{G} = Z_p$, $p \neq 0$, p prime, $A \subset \mathfrak{M}^n$, $\Lambda^k(A) = 0$, and let $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$ be a minimal compact set, and suppose $N < \infty$. Then:

I. If $p = 2$, then

$$\dim \mathfrak{L} + \dim \mathfrak{L}' \leq N \leq \dim H_k(\mathfrak{M}^n; Z_2) + \dim H_{k-1}(A \cup Z; Z_2).$$

II. If $p \neq 2$, then

$$\dim \mathfrak{L} + \dim \mathfrak{L}' \leq N_0 \leq \dim H_k(\mathfrak{M}^n; Z_p) + \dim H_{k-1}(A \cup Z; Z_p).$$

Corollary 1. Suppose that all the assumptions of Theorem 2 are fulfilled, and suppose additionally that $\mathfrak{L}' = 0$ and $\Lambda^{k-1}(Z) = 0$. Then for $p = 2$

$$N = \dim H_{k-1}(A; Z_2);$$

for $p \neq 2$

$$N_0 = \dim H_{k-1}(A; Z_p).$$

We emphasize that the numbers N and N_0 refer to different minimal compact sets corresponding to different coefficient groups.

5. We shall now consider the class $O^k(\emptyset, 0, \mathfrak{L}')$, $\mathfrak{L}' \neq 0$; a minimal compact set from this class will be denoted by Y_0^m (the letter m indicates minimality). Of all minimal compact sets $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$, the compact sets Y_0^m are apparently arranged most simply, since they are not obliged to span any boundary $A \subset \mathfrak{M}^n$. By Z, as usual, we denote the set of singular points of the compact set Y_0^m; it is known (see ([4])) that always $\Lambda^k(Z) = 0$. It turns out that the presence in the compact set Y_0^m of certain homological properties entails the appearance of a set Z of singular points of the greatest possible (in the sense of dimension) positive measure.

Theorem 3. Let $\mathfrak{M}^n$ be a compact, closed Riemannian manifold, $\mathfrak{M}^n \in C^4$, $\mathfrak{G} = Z_p$, $p \neq 0, 2$, p prime; $Y_0^m \in O^k(\emptyset, 0, \mathfrak{L}')$, $\mathfrak{L}' \neq 0$. Then, if $N < \infty$ and

$$\dim H_k(Y_0^m; Z_2) \neq \dim H_k(Y_0^m; Z_p),$$

then $\Lambda^{k-1}(Z) > 0$, and at least one of the groups

$$H_{k-1}(Z; Z_2), \qquad H_{k-1}(Z; Z_p)$$

is nontrivial. Moreover, all components Π_i are Z_p-orientable.

Definition 3. We shall say that a subgroup $\mathfrak{L}' \subset H_k(\mathfrak{M}^n; \mathfrak{G})$ admits an exact minimal realization if

$$\mathfrak{L}' \cong H_k(Y_0^m; \mathfrak{G}),$$

where $Y_0^m \in O^k(\emptyset, 0, \mathfrak{L}')$.

In the group $H_k(\mathfrak{M}^n; Z_p)$ one can always choose a basis consisting of elements $e_1, e_2, \ldots, e_r$, $r = \dim H_k(\mathfrak{M}^n; Z_p)$, admitting an exact minimal realization. Let $Y_{0j}^m \in O^k(\emptyset, 0, \{e_j\})$ be the minimal carriers of the one-dimensional subgroups $\{e_j\}$, $1 \leq j \leq r$.

Corollary 2. Let $\mathfrak{G} = Z_p$, $p \neq 0, 2$, p prime, and let $N < \infty$. Then, if $\dim H_k(Y_{0j}^m; Z_p) \neq 1$, then $\Lambda^{k-1}(Z) > 0$ and at least one of the groups $H_{k-1}(Z; Z_2)$, $H_{k-1}(Z; Z_p)$ is nontrivial.

Theorem 3 and Corollary 2 admit a transparent geometric interpretation. Roughly speaking, if an integral cycle is realized by means of a closed orientable manifold V^k, then $\dim H_k(V^k; Z_p) = \dim H_k(V^k; Z_2)$ and the set of singular points is empty. On the other hand, cycles of order p are arranged as CW-complexes $V^{k-1} \cup_f W^k$, where $\partial W^k = V^{k-1}$, and f is a mapping $V^{k-1} \to V^{k-1}$ of degree p (the manifold W^k is attached to V^{k-1} by this mapping), which leads to the inequality $1 = \dim H_k(Y_0^m; Z_p) \neq \dim H_k(Y_0^m; Z_2) = 0$ and to the appearance of a set Z of singular points ($Z = V^{k-1}$) for $p \neq 2$, and moreover

$\Lambda^{k-1}(Z) > 0$. It is also clear that for $p = 2$ the set Z may be empty; for example, $Z = \emptyset$ for the projective space RP^k, where $V^{k-1} = RP^{k-1}$, $W^k = D^k$.

6. Consider the special case when the compact set A is a $(k-1)$-dimensional sphere S^{k-1} embedded in $\mathfrak{M}^n$, and assume that the embedding is such that the class $N^k(A)$ is nonempty (see the definition in ([4])). As the coefficient group $\mathfrak{G}$ take the group $U = R^1(\text{mod}1)$. Such a fixed topological structure of the boundary allows one to obtain certain information of purely metric character concerning the structure of the minimal compact set $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$. We note that with each embedding $S^{k-1} \to \mathfrak{M}^n$ two numbers are associated: d' and $\tilde{d}$, where $d' = \inf \Lambda^k(X \setminus A)$, $X \in R^k(A)$; $\tilde{d} = \inf \Lambda^k(X \setminus A)$, $X \in N^k(A)$ (see ([4])); assume that $\tilde{d} < \infty$. Since in our case $\Lambda^k(A) = 0$, instead of the class $R^k(A)$ one may consider the broader class $R^k(\emptyset) \supset R^k(A)$, with $\inf \Lambda^k(X \setminus A) = d'$, where $X \in R^k(\emptyset)$. This means that the number d' is in fact determined only by the manifold $\mathfrak{M}^n$, namely: it is the Hausdorff measure of the "least" k-dimensional cycle in $\mathfrak{M}^n$. The number $\tilde{d}$ is the Hausdorff measure of the "least" film spanning a nontrivial subgroup in the group $H_{k-1}(S^{k-1}; U)$. Consider the decomposition of the submanifold $(X_0 \setminus A) \setminus Z$ into connected differentiable submanifolds Π_i (the number of components may also be infinite). The question arises: how "massive," in the sense of the measure Λ^k, can the individual components Π_i be.

Theorem 4. Let $\mathfrak{M}^n$ contain the compact set $A = S^{k-1}$, $X_0 \in O^k(A, \mathfrak{L}, \mathfrak{L}')$, where $\mathfrak{L} \neq 0$, $\mathfrak{L}' = U$; $d = \Lambda^k(X_0 \setminus A)$; $\Delta = \max(d - d', d - \tilde{d})$. Then $\Delta > 0$ and

$$\sup_i \Lambda^k(\Pi_i) \leq \Delta.$$

Remark. In the general case the estimate obtained in Theorem 4 cannot be improved: it is easy to indicate an example in which the value is attained for some $i = i_0$.

7. In Reifenberg' s paper ([1]) an existence theorem was proved for a minimal compact set in the class $\mathfrak{G}^* = N^k(S^{k-1})$, and, what is especially interesting, the compact sets $X \in \mathfrak{G}^*$ were described in terms of retraction mappings. Consider a compact pair (X, A), where $A = S^{k-1}$, coefficient group $\mathfrak{G} = U$. Reifenberg introduced two definitions of a boundary (see ([1])). We shall say that $A = \partial^R(X)$ if there does not exist a $(k-1)$-dimensional set $A^* \subset X$, $A^* \supset A$, such that $A^* = \operatorname{retr} X$ ($A^* = \operatorname{retr} X$ if there exists a continuous mapping $f : X \to A^*$ such that $f(A^*) \equiv A^*$). We shall say that $A = \partial^{\mathfrak{L}A}(X)$ if $\mathfrak{L} = b(X, A) \neq 0$ (see the definition in ([1])). From Hopf' s extension theorem it follows that these two definitions are equivalent. In ([1]) Reifenberg put forward the hypothesis that this equivalence also holds in the case when $A = \mathfrak{M}^{k-1}$ is an arbitrary compact closed orientable manifold. The following example shows that this hypothesis is false. Put $A = S^1 \times S^{k-2}$;

then $\pi_{k-1}(A) = Z_2$ for $k \geq 5$. Let $a \in [a] \in Z_2$ be a continuous mapping $S^{k-1} \to A$ corresponding to the generator $[a]$. Put $X = A \cup_a D^k$, attaching to A a k-dimensional disk D^k by the mapping a. It is clear that $A = \partial^R(X)$, but $A \neq \partial^{\mathfrak{L}A}(A)$ for any nontrivial subgroup $\mathfrak{L}$.

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Note: Figure translations are in progress. See original paper for figures.