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**Abstract**

**Full Text**

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*THEORY OF ELASTICITY*

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## INVESTIGATION OF THE PLASTICITY ZONE UNDER AFTEREFFECT FOR CERTAIN BODIES WITH CENTRAL AND AXIAL SYMMETRY

*(Presented by Academician Yu. N. Rabotnov on 8 VI 1970)*

Consider a hollow sphere of outer radius  $b$  and inner radius  $a$ , acted upon by an external  $q$  ( $q > 0$ ) and an internal  $p$  ( $p > 0$ ) uniform pressure. Suppose that, under the influence of the prescribed system of forces, at the instant  $t = 0$  a plastic zone arises in the sphere, completely embracing its inner surface. Since the material of the sphere is assumed to possess elastic-plastic properties with aftereffect, the radius of the plastic zone with hardening  $R(t)$ , for  $t > 0$ , will vary with time.

We shall investigate the quantity  $R(t)$  as a function of time  $t$ . We shall start from a physical equation of the form

$$\sigma_i = 3a_{2t}\varepsilon_i[1 - \omega(\varepsilon_i)], \quad (1)$$

where  $\sigma_i$  and  $\varepsilon_i$  are the intensities of stresses and strains, respectively <sup>(1)</sup>,  $a_{2t} = a_{20}(1 - R_2^*)$  is the operator shear modulus,  $a_{20}$  is the instantaneous shear modulus, and  $R_2^*$  is an integral operator with kernel  $R_2(t, \tau)$ . When acting on some function of time it has the form

$$a_{2t}\xi(t) = a_{20} \left[ \xi(t) - \int_0^t R_2(t, \tau)\xi(\tau) d\tau \right].$$

Here the function  $\omega$  is determined in accordance with <sup>(1)</sup>. In the case of linear hardening,

$$\sigma_i = 3a_{2t}\varepsilon_i \quad \text{for } \sigma_i < \sigma_s, \quad (2)$$

$$\sigma_i = a_{3t}\varepsilon_i + \varphi(t) \quad \text{for } \sigma_i \geq \sigma_s. \quad (3)$$

Here  $\varphi(t) = [(3a_{2t} - a_{3t})/3a_{2t}]\sigma_s$ ;  $a_{3t}$  is the operator hardening modulus, constructed analogously to the operator  $a_{2t}$  with kernel  $R_3(t, \tau)$ ;  $\sigma_s$  is the yield limit of the material. Taking into account that for a sphere <sup>(1)</sup>  $\sigma_i = (\sigma_\theta - \sigma_r)\chi$ ;  $\varepsilon_i = 2/3(\varepsilon_\theta - \varepsilon_r)\chi$ , from (3) we obtain

$$(\sigma_\theta - \sigma_r)\chi = 2/3a_{3t}(\varepsilon_\theta - \varepsilon_r)\chi + \varphi(t). \quad (4)$$

Here  $\sigma_r$  and  $\sigma_\theta$ ;  $\varepsilon_r$  and  $\varepsilon_\theta$  are the radial and tangential components of stresses and strains, respectively;  $u$  is the radial displacement of a point on the sphere;  $\chi = \text{sign } u$ .

To relation (4) there is adjoined the linear dilatational equation in operator form

$$\sigma = a_{1t}\theta, \quad (5)$$

where the operator  $a_{1t} = a_{10}(1 - R_1^*)$ ;  $3\sigma = 2\sigma_\theta + \sigma_r$ ;  $\theta = 2\varepsilon_\theta + \varepsilon_r$ .

From the system of equations (4) and (5) it follows that

$$\begin{aligned} \sigma_r &= a_{1t}(2u/r + du/dr) - 4/9a_{3t}(u/r - du/dr) - 2/3\chi\varphi(t), \\ \sigma_\theta &= a_{1t}(2u/r + du/dr) + 2/9a_{3t}(u/r - du/dr) + 1/3\chi\varphi(t). \end{aligned} \quad (6)$$

Substituting  $\sigma_r$  and  $\sigma_\theta$  from (6) into the equilibrium equation (1), we obtain an integro-differential equation in displacements

$$(a_{1t} + \mu_t)D\{u\} = \frac{2\varphi(t)}{r}\chi \quad (\mu_t = 4/9a_{3t}), \quad (7)$$

where the differential operator is

$$D\{u\} = \frac{d^2u}{dr^2} + 2\frac{1}{r}\frac{du}{dr} - 2\frac{u}{r^2}.$$

The general solution of equation (7) is found in two stages. We shall have

$$u = A(t)r + B(t)/r^2 + 1/3K(t)\ln r. \quad (8)$$

Here the known function  $K(t) = (a_{1t} + \mu_t)^{-1}2\varphi(t)\chi$ . The functions  $A(t)$  and  $B(t)$  are to be determined. Substituting into formulas (6), in place of the displacement  $u(r, t)$ , the expression found for it, we obtain

$$\sigma_r = 3a_{1t}A(t) - 3\mu_t B(t)/r^3 + a_{1t}K(t)\ln r + 1/3(a_{1t} + \mu_t)K(t) - 2/3\chi\varphi(t), \quad (9)$$

$$\sigma_\theta = 3a_{1t}A(t) + {}^3/2\mu_t B(t)/r^3 + a_{1t}K(t) \ln r + {}^1/3(a_{1t} - \mu_t/2)K(t) + {}^1/3\chi\varphi(t).$$

The expressions for the stress components  $\sigma_r^y$  and  $\sigma_\theta^y$  in the elastic zone are determined according to the scheme adopted above, with the original physical relation of the form (2). We shall have

$$\sigma_r^y = 3a_{1t}C_1 - 4a_{2t}C_2/r^3, \quad \sigma_\theta^y = 3a_{1t}C_1 + 2a_{2t}C_2/r^3. \quad (10)$$

Here  $C_1(t)$  and  $C_2(t)$  are functions of time to be determined. From the condition  $\sigma_i|_{r=R(t)} = \sigma_s$ , taking into account expressions (10), it follows that

$$6C_2(t) = R^3\chi\sigma_s a_{2t}^{-1}. \quad (11)$$

To determine the unknown function  $C_1(t)$ , we use the boundary condition  $\sigma_r^y|_{r=b} = -q$ . We obtain

$$C_1(t) = {}^2/9\chi m^3 a_{2t} a_{1t}^{-1} - \frac{R^3}{a^3} \sigma_s a_{2t}^{-1} - {}^2/9 a_{1t}^{-1} q,$$

where  $m = b/a$ . Taking into account the condition of equality of the corresponding stresses at the boundary of the elastic and plastic regions, we obtain a system of equations whose solution gives

$$9B(t) = \mu_t^{-1} R^3(t) [\chi\sigma_s + {}^1/2\mu_t K(t) - \chi\varphi(t)], \quad (12)$$

$$9a_{1t}A(t) = 9a_{1t}C_1(t) - 3a_{1t}K(t) \ln R(t) - a_{1t}K(t).$$

Using the condition on the inner contour  $\sigma_r^n|_{r=a} = -p$  and taking into account relations (11) and (12), we obtain a nonlinear integral equation for finding the sought function  $R(t)$ . We shall have

$$\alpha_t z - \beta_t \ln z = F(t), \quad (13)$$

where

$$\alpha_t z = {}^2/3\chi \frac{\sigma_s}{m^3} a_{2t} z a_{2t}^{-1} - {}^2/3\chi\sigma_s z - {}^1/3z\mu_t K(t) + {}^2/3\chi z\varphi(t);$$

$$\beta_t = {}^1/3 a_{1t} K(t); \quad F(t) = -p + q - {}^1/3\mu_t K(t) + {}^2/3\chi\varphi(t); \quad z = R^3/a^3.$$

For what follows it is expedient to reduce equation (13) to the form

$$z = z_0 \exp(-\chi_t z), \quad (14)$$

where

$$\chi_t = -\alpha_t \beta_t^{-1}, \quad z_0 = \exp[-\beta_t^{-1} F(t)].$$

We shall solve equation (14) by the method of successive approximations. As the zeroth approximation we take  $z_0$ . Then successively—

the approximations are determined as follows

$$z_n = z_0 \exp(-\varkappa_t z_{n-1}).$$

Let us establish a two-sided estimate of the solution  $z(t) = \lim_n z_n(t)$  of equation (14).

First consider the first approximation

$$z_1 = z_0 \exp(-\varkappa_t z_0). \quad (15)$$

Taking into account that the operator  $\varkappa_t = (m^3 - m^3 \chi_t - 1) \chi_t^{-1}$ , where  $3\chi_t = (3a_{2t} - a_{3t}) \cdot a_{1t} a_{2t}^{-1} (a_{1t} + \mu_t)^{-1}$ , and assuming  $m^3(1 - \chi_t) > 1$ , we find that  $\varkappa_t \cdot 1 > 0$ . Applying further the mean-value theorem in (15), we obtain

$$z_1(t) = z_0(t) \exp[-z_0(\theta t)N] \quad (N = \varkappa_t \cdot 1; \quad 0 < \theta < 1). \quad (16)$$

Let  $\chi_\infty > \chi_0$ . The latter holds at least in the case of absence of dilatational aftereffect and when the piecewise-linear approximation is realized with respect to Rabotnov' s nonlinear physical dependence <sup>(2)</sup>. Then, taking into account that  $z_0 = \exp[(p - q - \chi_t) \chi_t^{-1}]$ , we find  $z_0(\infty) < z_0(t) < z_0(0)$ . It follows from (16) that

$$z_1(t) < z_0(t) \exp[-z_0(t)N].$$

Since  $z_0(t) > 1$ , then  $\ln z_0(t) < z_0(t)$ . Therefore  $1 < z_1 < z_0^{1-N}$ . Similarly, for the second approximation:  $1 < z_2 < z_0^{1-N+N^2}$ . For the  $n$ -th approximation we shall have the estimate

$$1 < z_n < z_0^{(1+(-1)^n \eta^{n+1})/(1+N)}. \quad (17)$$

From the expression for  $\nu_t$  it follows that the inequalities  $0 < N < 1$  hold under the condition

$$m^3(1 - \chi_t) < m^3 < (1 + \chi_t)(1 - \chi_t)^{-1}. \quad (18)$$

In this case it follows from (17) that

$$1 < \lim_{n \rightarrow \infty} z_n < z_0^{1/(1+N)}. \quad (19)$$

If conditions (18) are not satisfied, the solution of the problem is realized by introducing the quantity  $k = aN$ , where  $0 < a < N^{-1}$ . Then  $0 < k < 1$ , and inequality (19) retains its meaning also for  $N > 1$ .

A problem of determining the plasticity zone in an infinite cylinder is also reduced to an equation of the form (13). Thus, in the case of a cylinder reinforced by an elastic shell, we have

$$\alpha_t z = 4(\delta_t z f(t) - a_{1t} \xi_t z); \quad \beta_t = -\frac{2(\lambda_t + \delta_t)}{\lambda + 2\delta_t} \psi(t),$$

$$F(t) = 2 \left( p + a_{1t} \frac{\sigma_s}{3a_{2t}\gamma} - \frac{\delta_t - \sqrt{3}\gamma a_{2t}}{\delta_t + a_{1t}} \right),$$

where

$$\delta_t = \frac{\gamma}{\sqrt{3}} a_{2t}; \quad f(t) = \frac{\sigma_s}{a_{1t} + \delta_t} - \frac{\sqrt{3}a_{1t} + 3\gamma a_{2t}}{6\sqrt{3}\gamma a_{2t}},$$

$$\xi_t z = \frac{6\alpha\gamma a_{2t} - \sqrt{3}b}{\sqrt{3}(b + 2\alpha a_{1t})m^2} z - \frac{\sigma_s}{6\gamma a_{2t}}; \quad \alpha = \frac{\rho^2(1 - \nu^{(0)2})}{E^{(0)}h},$$

$\rho, h$  are the radius and thickness of the shell,  $E^{(0)}, \nu^{(0)}$  are the modulus of elasticity and Poisson's ratio of the shell material,  $\lambda_t = a_{1t} - \delta_t$ ;  $\psi(t) = \frac{1}{\sqrt{3}}\varphi(t)$ ,  $\gamma$  is the approximating coefficient according to A. A. Ilyushin<sup>(3)</sup>, which figures in the approximate representation  $\varepsilon_i = \gamma(\varepsilon_\theta - \varepsilon_r)$ .

In the case of a cylinder not placed in a shell, the corresponding two-sided estimate of the solution of an equation of the form (13) will hold irrespective of the fulfillment of conditions of type (18), whereas for a reinforced cylinder such an estimate can be established only under certain restrictions concerning the thickness of the cylinder, the elastic characteristics of the shell material, and the rheological constants that ensure convergence of successive approximations to the desired solution.

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## References

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- <sup>3</sup> A. A. Ilyushin, *PMM*, **10**, issue 3 (1946).

*Note: Figure translations are in progress. See original paper for figures.*

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