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Abstract

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MATHEMATICS

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A GENERAL FIRST MAIN THEOREM OF VALUE DISTRIBUTION THEORY

(Presented by Academician P. S. Aleksandrov on 16 I 1970)

In recent years a number of works have appeared devoted to proving that, under certain conditions imposed on analytic mappings of manifolds, almost all values are, in a certain sense, distributed uniformly. At the same time, very substantial restrictions are imposed on the target manifold. Thus, in works ($^{1-3}$) mappings into the projective space CP^n are considered; in (^{4}), into a homogeneous Kähler manifold. Finally, in ($^{5-7}$) mappings into arbitrary Kähler manifolds are considered.

In the present note we obtain the same results without any restrictions on the target manifold.

The following theorem lies at the basis of our considerations.

Theorem 1. Let M be a compact complex n-dimensional manifold and let ω be a form of type (n, n) defining on M a volume element, with

$$\int_M \omega = 1.$$

Then on M one can define such a family of forms λ_a, depending on the parameter $a \in M$, that:

1. The form λ_a is defined and belongs to the class C^∞ everywhere except at the point a.
2. In a neighborhood of the point a, the form λ_a is representable in the form $\varkappa\mu$, where μ is a form of class C^∞, and $\varkappa$ is a function which in suitable local coordinates has the form $|z|^{-2n+2} \log |z|$ (it is assumed that a is the origin of the coordinates).
3. The forms λ_a are positive real forms of type $(n-1, n-1)$, i.e., written in local coordinates, the coefficients of these forms form a positive definite Hermitian matrix.
4. λ_a depends continuously on a.

5. Outside the point a the basic equality holds

$$dd^c\lambda_a = \omega.$$

Remark. For the projective space CP^n such a form was constructed by Levine (^{2}). For a homogeneous Kähler manifold the existence of such a form was established by Hirschfelder (^{4}), and for an arbitrary Kähler manifold by Wu (^{5}).

The proof of this theorem is based on the following lemma.

Lemma. On a compact complex n-dimensional manifold there exists a strictly positive form μ of type $(n-1, n-1)$ such that

$$dd^c\mu = 0.$$

Proof of the lemma. Let μ_0 be an arbitrary strictly positive form of type $(n-1, n-1)$, and let ω be an arbitrary strictly positive form of type (n, n). Then for any function f on M there is uniquely determined a function φ such that $dd^c(f\mu_0) = \varphi\omega$. The correspondence $f \mapsto \varphi$, obviously, defines on M an elliptic operator A. The adjoint-

adjoint operator A^*, as is easy to see, is defined by the correspondence $f \mapsto \varphi$, where φ is such a function that $dd^c f \wedge \mu = \varphi\omega$.

From the form of the operator A^* and the general theory of elliptic equations it follows that the equation $A^*f = q$ has no solutions if the function q is nonnegative and is not identically zero. Hence it is easy to obtain that the equation $Af = 0$ has a nonnegative solution which vanishes only on a nowhere dense set.

Let x_0 be an arbitrary point of the manifold M. Choose the form μ_0 so that in a neighborhood of the point x_0 the corresponding operator A is written in local coordinates as the Laplacian $\sum \partial^2/\partial z_i \partial \bar{z}_i$. If f is the solution of the equation $Af = 0$ discussed above, then $f(x_0) > 0$ by virtue of the mean-value theorem for harmonic functions. The form $f\mu_0$ is everywhere nonnegative and is positive definite in a neighborhood of the point x_0. Moreover, $dd^c f\mu_0 = 0$. Covering the manifold M by a finite number of corresponding neighborhoods and summing the corresponding forms, we obtain the desired form μ.

Proof of Theorem 1. Let B_r be the ball in C^n of radius r, and let $\varphi_1, \dots, \varphi_k$ be mappings of the ball B_1 into M such that $\varphi_i(B_1)$ are coordinate neighborhoods and define a covering. Choose ε so that the sets $\varphi_i(B_{1-\varepsilon})$ still form a covering. For each point $z \in B_{1-\varepsilon}$ one can construct a form λ'_z whose support is the ball of radius ε with center at z, such that conditions 1-4 are satisfied, and, moreover, $dd^c\lambda'_z$ can be extended to a C^∞-form at the point z, and

$$\int_{C^n} dd^c\lambda'_z = 1.$$

Let $\psi_1, \dots, \psi_k$ be a partition of unity on the manifold M corresponding to the covering $\varphi_1(B_{1-\varepsilon}), \dots, \varphi_k(B_{1-\varepsilon})$. Then the form $\lambda''_a = \sum \psi_i(a) \times (\varphi_i^{-1})^* \lambda_{\varphi_i^{-1}(a)}$ satisfies conditions 1-4, and, moreover, $dd^c\lambda''_a$ extends at a to a C^∞-form, and

$$\int_M dd^c\lambda''_a = 1.$$

We shall use the form μ constructed in the lemma. From the general theory of elliptic equations it follows that the equation $dd^c f\mu = \omega - dd^c\lambda''_a$ has a regular solution f, depending continuously on a, since the right-hand side is orthogonal to all solutions of the adjoint equation (these are constants). If now K is chosen so that the inequality $f + K > 0$ is satisfied, then the form $\lambda_a = \lambda''_a + (f + K)\mu$ will be the desired one.

Now consider a complex n-dimensional manifold X and suppose that an exhaustion ψ is given on X, i.e., a nonnegative real function such that the inverse images of compact sets are compact and the critical points of this function are isolated. Let f be a holomorphic mapping of the manifold X into a compact complex n-dimensional manifold M. Define on M the forms ω and λ_a discussed in Theorem 1, and introduce the following notation. By D_t we denote the set of those $x \in X$ for which $\psi(x) \leq t$. If a point $a \in M$ is such that the set $f^{-1}(a) \cap D_t$ is discrete, then by $n(t, a)$ we denote the number of points in $f^{-1}(a) \cap D_t$, counted with multiplicities. Standard arguments using Stokes' formula lead to the following result.

Theorem 2 (the nonintegrated first main theorem).

$$\int_{D_t} f^*\omega = n(t, a) + \int_{\partial D_t} f^*(d^c\lambda_a).$$

Integration of this equality with respect to t, after simple transformations, leads to the following theorem.

Theorem 3 (the first main theorem).

$$N(r, a) = T(r) + m(r_0, a) - m(r, a) + \Delta(r, a).$$

Here the following notation has been used:

$$N(r, a) = \int_{r_0}^{r} n(t, a)\, dt, \qquad T(r) = \int_{r_0}^{r} \left(\int_{D_t} f^*\omega \right) dt,$$

$$m(r, a) = \int_{\partial D_r} d^c\psi \wedge f^*\lambda_a, \qquad \Delta(r, a) = \int_{D_t} dd^c\psi \wedge f^*\lambda_a.$$

Imposing various conditions on the exhaustion ψ and on the mapping f, one can, just as was done in the papers (^{3},^{6}), obtain various theorems on the equidistribution of values. If, for example, the exhaustion ψ is pseudoconcave (i.e., the form $dd^c\psi$ is negatively semidefinite), then $m(r,a) > 0$, $\Delta(r,a) < 0$, $N(r,a) \leq T(r) + \text{const}$, and therefore, for every $a \in M$, the relation $\overline{\lim}\, N(r,a)/T(r) \leq 1$ holds. From the fact that

$$\int_M N(r,a)\omega = T(r),$$

it is now easy to obtain that, for all $a \in M$, except for a set of measure zero, the relation $\overline{\lim}\, N(r,a)/T(r) = 1$ holds. The special case in which the complex space CP^n is taken as M was obtained in (^{8}), where this result is formulated in other terms.

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Note: Figure translations are in progress. See original paper for figures.

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