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Abstract

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ON THE SUBDIFFERENTIAL OF A COMPOSITE FUNCTIONAL

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In recent years a number of works have appeared in which subdifferentials of convex functionals of various types were described (see ([1–10])). The popularity of this topic is connected with the important role that subdifferentials play in the theory of nonsmooth extremal problems, as was first observed by A. Ya. Dubovitskii and A. A. Milyutin ([1]). Most of the functionals considered in the cited works were composite, i.e., they represented a composition of convex mappings and concrete convex functionals.

In the present note the subdifferential of a composite functional of general form is described. We restrict ourselves to the case of Banach spaces, although the results given below are also valid in a more general situation. We shall use the concepts of the theory of semiordered linear spaces of L. V. Kantorovich and employ terminology that for the most part coincides with that adopted in ([11]).

Let G be an open convex set in a real Banach space X; let Y be a Banach lattice (a KB-lineal in the terminology of ([11])). A mapping $F : G \to Y$ is called **convex** if

$$F[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda F(x_1) + (1 - \lambda)F(x_2)$$

for any $x_1, x_2 \in G$ and any $0 < \lambda < 1$.

Let U be an open convex set in Y; $F : G \to U$ a continuous convex mapping; $x_0 \in G$; φ a convex functional on U. The simplest examples show that these conditions are insufficient for the convexity of the composite functional $f(x) = \varphi[F(x)]$. The functional f will be convex if, in addition, one requires monotonicity of φ. The following proposition shows that in this case f will also be continuous.

Proposition 1. *Let φ be a monotone convex functional on an open subset U of a Banach lattice Y. Then it is continuous.*

In what follows the functional φ is assumed to be convex and monotone. Let us describe the subdifferential $\partial f(x_0)$.

We shall call the **subdifferential of the mapping** F at the point x_0 the set $\partial F(x_0)$ of all continuous linear mappings $A : X \to Y$ satisfying the inequality $Ax - Ax_0 \leq F(x) - F(x_0)$ for every $x \in G$. When Y is the number line, we obtain the well-known definition of the subdifferential of a convex functional.

Let Y be a conditionally complete Banach lattice (a KN-space in the terminology of ([11])). It is said that Y has **property (A)** if for every sequence $(y_n) \subset Y$ it follows from $y_n \downarrow 0$ that $\|y_n\| \to 0$ as $n \to \infty$. Many Banach lattices have property (A), in particular L_p for $1 \leq p < \infty$, and it plays an important role in the theory of semiordered linear spaces (see ([11])). The following result is apparently new.

Proposition 2. Let Y be a conditionally complete Banach lattice. Then property (A) is equivalent to weak relative bicompactness* of the order-bounded subsets of Y.

Theorem 1. Let Y be a conditionally complete Banach lattice possessing property (A). Then $\partial F(x_0)$ is a nonempty convex set, bicompact in the weak operator topology.

Recall that the weak operator topology on the space $H(X, Y)$ of continuous linear mappings $A : X \to Y$ is defined by the following base of neighborhoods of zero

$$W\{x_1, \dots, x_m; \lambda_1, \dots, \lambda_n\} = \{A \in H(X, Y) : |\lambda_k(Ax_j)| \leq 1,\ j = 1, \dots, m;\ k = 1, \dots, n\},$$

where $\{x_1, \dots, x_m\}$ and $\{\lambda_1, \dots, \lambda_n\}$ range over all possible finite sets of elements of X and Y', respectively.

Remark. If Y is an arbitrary Banach lattice, the set $\partial F(x_0)$ may turn out to be empty.

We formulate the main result.

Theorem 2. Let Y be a conditionally complete Banach lattice possessing property (A), and let $f(x) = \varphi[F(x)]$. Then $\partial f(x_0)$ is the set of all possible functionals $l(x)$ representable in the form $l(x) = \lambda(Ax)$, where $\lambda \in \partial\varphi[F(x_0)]$, $A \in \partial F(x_0)$.

Remark. For an arbitrary Banach lattice Y, this theorem is, generally speaking, false.

It would be interesting to extend Theorem 2 to the case when x_0 is a boundary point of the domain of definition of f, analogously to how this was done in papers ([8–10]) for functionals φ of the type of an integral and of taking a maximum.

In conclusion we give one useful, though simple, proposition concerning arbitrary Banach lattices.

Proposition 3. Every linear mapping of a Banach space into a Banach lattice is continuous.

For the proof one must use Banach' s closed graph theorem and one property of convergent sequences in a Banach lattice (([11]), Theorem VII. 2. 1, p. 196).

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* A set is called **relatively bicompact** if its closure is bicompact.