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**Abstract**

**Full Text**

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**MATHEMATICS**

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**APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS OF MANY VARIABLES BY GENERALIZED POLYNOMIALS (FINITE LINEAR SUPERPOSITIONS OF FUNCTIONS OF A SMALLER NUMBER OF VARIABLES)**

*(Presented by Academician A. N. Tikhonov, October 31, 1969)*

The problem of representing, exactly or approximately, continuous and differentiable functions of  $n$  variables defined on the unit cube  $\bar{\Omega}^{(n)}$  in the form of a finite superposition of functions of a smaller number of variables is very important both theoretically and from the point of view of applications to the constructive theory of functions, the general theory of approximate methods, and other problems of analysis. This question received its most complete theoretical solution in the second half of the 1950s in a number of works of A. N. Kolmogorov's school (see, for example, the survey <sup>(1)</sup>). The strongest result here belongs to A. N. Kolmogorov <sup>(2)</sup>, according to which any continuous function  $f$  on  $\bar{\Omega}^{(n)}$  is exactly representable by  $2n + 1$  functions of one variable; moreover, the argument of each of these functions is the sum of  $n$  standard (i.e., not depending on  $f$ ) functions, also of only one variable. Of course, it does not follow from this that, in order to study the properties of functions of  $n$  variables, it is sufficient to restrict oneself to studying the properties of functions of only one variable (although the construction obtained in <sup>(2)</sup> is very convenient—namely, the function  $f$  depends linearly on the  $2n + 1$  functions determining  $f$ ), since, as noted at the end of the survey <sup>(1)</sup>, for practical purposes such representations are apparently useless, since they use essentially nonsmooth functions, such as the Weierstrass function.\* Thus, the representation of  $f$  in the form of a finite superposition of functions of a smaller number of variables whose differentiability properties are no worse than the differentiability properties of  $f$  must be approximate, and such representations, convenient for applications, should naturally be sought first of all by means of a set of sufficiently smooth standard functions ensuring the existence and uniqueness of the approximating function, whose form, on the one hand, would depend linearly on the functions

(or constants) determining  $f$ , and, on the other hand, would possess the best approximation properties in one or another class of functions in comparison with any other linear form using the same set of standard functions. This note is devoted to the consideration of several questions related to this problem.

Let  $M_1, \dots, M_n$  be integers,  $x_1, \dots, x_n$  a Cartesian coordinate system;  $c_{m_i}^{(i)}(x_i) \in C[0, 1]$  ( $m_i = 1, \dots, M_i$ ) a given set of standard functions, linearly independent for each fixed number  $i \leq n$ . Let  $\bar{a}_n = \{a_1, \dots, a_n\}$  be an  $n$ -component set,  $\bar{a}_n - a_k = \{a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n\}$  an  $(n-1)$ -component subset of  $\bar{a}_n$ ;  $\bar{a}_{n,k}$  an arbitrary subset of  $\bar{a}_n$  with  $k$  components, and if  $\bar{a}_{n,k}$  and  $\bar{b}_{n,k}$  contain—

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\* See also the survey <sup>(3)</sup>, in which later results are also given—mostly negative ones—on the possibility of an exact representation of differentiable (or analytic) functions in the form of a finite superposition of differentiable (or analytic) functions of a smaller number of variables.

the components  $\bar{a}_n$  and  $\bar{b}_n$  with identical numbers, we write  $\overrightarrow{\bar{a}_{n,k}}$  and  $\overrightarrow{\bar{b}_{n,k}}$ .

The most general linear form of polynomials based on functions  $c_{m_i}^{(i)}$ , containing functions of no more than  $n-1$  variables, is, obviously,

$$F_{\bar{M}_n}^{(n-1)} = \sum_{i=1}^n \sum_{m_i=1}^{M_i} \tilde{F}_{m_i}^{(i)}(\bar{x}_n - x_i) c_{m_i}^{(i)}(x_i). \quad (1)$$

**Definition 1.** We shall call the polynomial (1) a **generalized polynomial of order  $\bar{M}_n$  and rank  $n-1$  relative to the system of functions  $c_{m_i}^{(i)}(x_i)$** . A generalized polynomial of order  $\bar{M}_n$  and rank  $k-1$  is formed from a generalized polynomial of order  $\bar{M}_n$  and rank  $k$  (notation  $F_{\bar{M}_n}^{(k)}$ ) after replacing, in each last function of  $k$  variables  $\overrightarrow{\bar{x}_{n,k}}$ , by a generalized polynomial of order  $\overrightarrow{\bar{M}_{n,k}}$  and rank  $k-1$ .

The polynomial  $F_{\bar{M}_n}^{(1)}$  may be interpreted as a natural generalization of the form

$$\sum_{i=1}^n \varphi_i(x_i), \quad (2)$$

in which all functions are determined from  $f$  and which is the simplest generalized polynomial of order  $\bar{M}_n = \{1, \dots, 1\}$  and rank 1. Polynomials of the form (2) were considered by A. N. Kolmogorov (see <sup>(1,4)</sup>), and also in <sup>(4-6)</sup>; polynomials  $F_{\bar{M}_n}^{(k)}$  with respect to a power system of functions—in <sup>(7,8)</sup>, partial sums of a Fourier series—in <sup>(9,10)</sup>. We note that the polynomial  $F_{\bar{M}_n}^{(n-1)}$  also generalizes the partial sums used in <sup>(11)</sup> for the approximation of a class of functions

with bounded mixed derivative and in <sup>(12)</sup> for the approximation of the class of functions of S. M. Nikol'skii <sup>(13)</sup>.

1. If  $F_{M_n}^{(k)}$  ( $0 \leq k \leq n - 1$ ) is a polynomial of best uniform approximation (b.u.a.) to a function  $f(\bar{x}_n) \in C(\bar{\Omega}^{(n)})$ , then theorems analogous to theorems for polynomials of b.u.a. of functions of one variable and polynomials of the form (2) of b.u.a. of functions of  $n$  variables hold for it.

**Theorem 1.** *If, for each  $i \leq n$ , the moduli of continuity of the functions  $c_{m_i}^{(i)}(x_i)$  are no worse than the  $i$ -th partial modulus of continuity of the function  $f(\bar{x}_n) \in C(\bar{\Omega}^{(n)})$ , then for  $f(\bar{x}_n)$  there exists a generalized polynomial  $F_{M_n}^{(k)}$  of b.u.a.; moreover, the functions of  $k$  variables  $\bar{x}_{n,k}$  forming this polynomial have partial moduli of continuity with respect to the variable  $x_i \in \bar{x}_{n,k}$  no worse than the  $i$ -th partial modulus of continuity of the function  $f(\bar{x}_n)$ .*

**Theorem 2** (an analogue of the theorem of A. N. Kolmogorov <sup>(14)</sup>). *In order that a generalized polynomial  $F_{M_n}^{(k)}$  be a polynomial of b.u.a. for a function  $f(\bar{x}_n) \in C(\bar{\Omega}^{(n)})$ , it is necessary and sufficient that, for every generalized polynomial  $G_{M_n^*}^{(k)} \in C(\bar{\Omega}^{(n)})$ , where the corresponding components of the sets  $\bar{M}_n^*$  and  $\bar{M}_n$  satisfy the condition  $M_i^* \leq M_i$ , the inequality*

$$\max_S \{G_{M_n^*}^{(k)}(f - F_{M_n}^{(k)})\} \geq 0,$$

hold, where  $S \subset \bar{\Omega}^{(n)}$  is the set of all those points of  $\bar{\Omega}^{(n)}$  at which

$$|f - F_{M_n}^{(k)}| = \max_{\bar{\Omega}^{(n)}} |f - F_{M_n}^{(k)}|.$$

2. Here we shall consider b.u.a. by generalized polynomials  $F_{M_n}^{(k)}$  of a class of functions representable by integral operators, for which the fundamental theorem of constructive theory can be formulated in terms of b.u.a. of functions of one variable.

Let  $\rho_1(x, t_1), \dots, \rho_n(x_n, t_n)$  be functions continuous in  $\bar{\Omega}^{(2)}$ ; let  $\Phi_s$  be sets of functions  $\varphi_s \in \Phi_s$  of  $s$  variables, summable in  $\bar{\Omega}^{(s)}$ , satisfying there the condition  $|\varphi_s| \leq 1$ . By  $D_{\rho_i}, D_{\bar{\rho}_{n,k}}$  we denote the classes of continuous functions representable, respectively, in the form of the integrals

$$\int_0^1 \rho_i(x_i, t_i) \varphi_1(t_i) dt_i; \quad \underbrace{\int_0^1 \dots \int_0^1}_{k \text{ times}} \prod \bar{\rho}_{n,k} \varphi_k(\bar{t}_{n,k}) \prod d\bar{t}_{n,k},$$

where the products in the second integral extend over all  $k$  components entering into  $\bar{\rho}_{n,k}$  and  $d\bar{t}_{n,k}$ .

**Theorem 3.** Let, for each  $i \leq n$ ,  $c_{m_i}^{(i)}(x_i) \in D_{\rho_i}$  and let  $P_{M_i}^{(i)}(x_i)$  be a polynomial of order not exceeding  $M_i$  with respect to this system. Suppose further that numbers  $B_{M_i}^{(i)} \leq B < \infty$  are given and the sequence of inequalities

$$\sup_{f_i \in D_{\rho_i}} \inf_{P_{M_i}^{(i)}} \max_{x_i \in [0,1]} |f_i - P_{M_i}^{(i)}| \leq B_{M_i}^{(i)}, \quad i = 1, \dots, n, \quad (3)$$

holds, and moreover for any  $f_i(x_i) \in D_{\rho_i}$  there is a polynomial  $P_{M_i}^{(i)}$  such that

$$|f_i - P_{M_i}^{(i)}| \leq B_{M_i}^{(i)} b_i(x_i), \quad (4)$$

where  $b_i(x_i) \in C[0, 1]$  does not depend on  $f_i(x_i)$ ;  $0 \leq b_i(x_i) \leq 1$ .

Then, for the n. r. p. class of functions  $D_{\bar{\rho}_n}$ , with generalized polynomials  $F_{\bar{M}_n}^{(k)}$ , the inequality

$$\sup_{f \in D_{\bar{\rho}_n}} \inf_{F_{\bar{M}_n}^{(k)}} \max_{\bar{x}_n \in \bar{\Omega}^{(n)}} |f - F_{\bar{M}_n}^{(k)}| \leq c \sum \prod \bar{E}_{n,k+1} \quad (3')$$

is valid; and for any  $f(\bar{x}_n) \in D_{\bar{\rho}_n}$  there is a polynomial  $F_{\bar{M}_n}^{(k)}$  such that

$$|f - F_{\bar{M}_n}^{(k)}| \leq c \sum \prod \bar{H}_{n,k+1}, \quad (4')$$

where (3') follows from (3), and (4') from (4). The constant  $c$ , which for  $k = n - 1$  may be taken equal to 1, does not depend on  $\bar{M}_n$  and  $\bar{x}_n$ . Further,  $\bar{E}_n = \{B_{M_1}^{(1)}, \dots, B_{M_n}^{(n)}\}$ ,  $\bar{H}_n = \{B_{M_1}^{(1)} b_1(x_1), \dots, B_{M_n}^{(n)} b_n(x_n)\}$ , and by  $\sum \prod \bar{r}_{n,k+1}$  is denoted the sum of all distinct products of  $k + 1$  components of the  $n$ -component set  $\bar{r}_n = \{r_1, \dots, r_n\}$ . The functions of  $k$  variables  $\bar{x}_{n,k}$  forming the polynomial  $F_{\bar{M}_n}^{(k)}$ , satisfying the inequality (4'), belong to the classes of functions  $D_{\bar{\rho}_{n,k}}$ .

**Remark 1.** If the quantities  $B_{M_i}^{(i)}$  are of the same order for any  $i \leq n$ , and moreover

$$\lim_{M_i \rightarrow \infty} B_{M_i}^{(i)} = 0,$$

then the generalized polynomial  $F_{\bar{M}_n}^{(k)}$ , n. r. p., possesses, on the entire class  $D_{\bar{\rho}_n}$ , better (with respect to the order of the upper bound) approximative properties than any linear polynomial with respect to the system  $c_{m_i}^{(i)}(x_i)$  containing arbitrary functions of  $k$  variables, if the number of these functions does not exceed

the number of all distinct functions of  $k$  variables entering into the polynomial  $F_{\overline{M}_n}^{(k)}$ .

**Remark 2.** The assertions of Theorem 3 and Remark 1, formulated in terms of the best approximation in  $L_r(\overline{\Omega}^{(n)})$  ( $r \geq 1$ ), remain valid if by  $\Phi_s$  and  $D_{\overline{\rho}_{n,k}}$  one understands, respectively, sets of  $L_r$ -summable functions with  $\|\varphi_s\|_{L_r} \leq 1$ .

3. If the functions of  $k$  variables forming the polynomial  $F_{\overline{M}_n}^{(k)}$  \* are determined from the interpolation conditions for the continuous function  $f(\bar{x}_n)$  on a set of the cube  $\overline{\Omega}^{(n)}$  of a special form, then for  $F_{\overline{M}_n}^{(k)}$  theorems analogous to Theorems 1, 3 hold.

**Definition 2.** An **interpolation lattice** (or lattice) **of order  $\overline{M}_n$  of rank  $n-1$**  will mean the aggregate  $\Gamma_{\overline{M}_n}^{(n-1)} \subset \overline{\Omega}^{(n)}$  of hyperplanes  $x_p = \text{const}$ , which for each number  $i \leq n$  contains  $M_i$  distinct hyperplanes parallel to the hyperplane  $x_i = 0$ . The lattice  $\Gamma_{\overline{M}_n}^{(k-1)}$  is formed from the lattice  $\Gamma_{\overline{M}_n}^{(k)}$  by deleting from  $\Gamma_{\overline{M}_n}^{(k)}$  all points that do not belong simultaneously to two hyperplanes of dimension  $k$  of the lattice  $\Gamma_{\overline{M}_n}^{(k)}$ .

**Theorem 4.** *If for each  $i \leq n$  the moduli of continuity of the system of functions  $c_{m_i}^{(i)}(x_i)$  are no worse than the  $i$ -th partial modulus of continuity of the function  $f(\bar{x}_n) \in C(\overline{\Omega}^{(n)})$ , then there exists a unique generalized polynomial  $F_{\overline{M}_n}^{(k)}$  with respect to this system, interpolating  $f(\bar{x}_n)$  on the lattice  $\Gamma_{\overline{M}_n}^{(k)}$ , and the functions of  $k$  variables  $\bar{x}_{n,k}$  forming this polynomial have partial moduli of continuity in the variable  $\bar{x}_i \in x_{n,k}$  no worse than the  $i$ -th partial modulus of continuity of the function  $f(\bar{x}_n)$ .*

**Theorem 5.** *The assertions of Theorem 3 and Remark 1 are also valid in the case when, in them, instead of the lower bound for  $P_{\overline{M}_i}^{(i)}$  and  $F_{\overline{M}_n}^{(k)}$ , one understands respectively the interpolation polynomials of the functions  $f_i(x_i)$  and  $f(\bar{x}_n)$  on the lattice  $\Gamma_{\overline{M}_i}^{(0)}$  (with  $\overline{M}_1 = \{M_i\}$ ) and the lattice  $\Gamma_{\overline{M}_n}^{(k)}$ .*

4. The generalized polynomials  $F_{\overline{M}_n}^{(k)}$  are convenient for constructing cubature formulas. Replacing the function  $f(\bar{x}_n)$  by some generalized polynomial (of best approximation, interpolation, a partial sum of a Fourier series, etc.), we reduce the problem of computing a cubature in an  $n$ -dimensional domain to computing a finite number of cubatures in domains of dimension not exceeding  $k$  ( $k < n$ ). On the other hand, for the error of such a cubature formula a remark of the type of Remark 1 is valid.
5. The polynomials  $F_{\overline{M}_n}^{(k)}$  can be used for constructing direct methods for solving equations of mathematical physics <sup>(15)</sup>. In this case, for example, when the desired solution belongs to the class of functions  $D_{\rho_n}$ , and the approximate solution  $F_{\overline{M}_n}^{(k)}$  is sought from the condition of interpolation of the equation on the lattice  $\Gamma_{\overline{M}_n}^{(k)}$  (such a method is naturally called

the method of lattice collocation), for estimating the error one can also formulate a remark of the type of Remark 1 (see also <sup>(15)</sup>, Remark 1).

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\* Here the system of functions  $c_{m_i}^{(i)}(x_i)$  is assumed to be a Chebyshev system on the interval  $[0, 1]$  for each number  $i \leq n$ .

*Note: Figure translations are in progress. See original paper for figures.*

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