THE SPECTRUM OF UNIFORM HOMOLOGIES

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THE SPECTRUM OF UNIFORM HOMOLOGIES

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In the paper ([4]) groups of uniform homologies were constructed. These groups, however, do not make it possible to distinguish even such nonhomeomorphic manifolds as $w = u^2 + v^2$ and $w = u^4 + v^4$ in E^3. Let us note that between these manifolds there is the following distinction. In the first of them take a cycle Z with support $w = c$ and an m-bounded chain X with support $|X|$ of minimal diameter. It is easy to see that $d(X)$, as a function of $d(Z)$ ($d(X)$ is the diameter of the set $|X|$), has order of growth equal to the order of growth of the function $\varphi(t) = t^2$. An analogous construction for the second manifold gives the order of growth of the function $\varphi(t) = t^4$.

In the present paper groups are constructed that reveal the indicated difference in orders of growth. To each order of growth there is assigned a group of uniform homologies of a metric space; moreover, with respect to the natural direction in the set of orders of growth, these groups form an inverse spectrum. In particular, for a function of the same order as $\varphi(t) = t$, on manifolds with a uniform metric we obtain the groups Q_r constructed earlier ([4]).

1. We shall need the following definitions.

A. A Riemannian manifold R^n is called a **manifold with a uniform metric** ([4]) if there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that for every point $x \in R^n$ there exists a mapping f of some neighborhood U_x of this point onto the Euclidean n-dimensional ball of unit radius, satisfying the condition

$$\gamma_1 \leq \rho(\bar{x}, \bar{\bar{x}})/\rho(f(\bar{x}), f(\bar{\bar{x}})) \leq \gamma_2, \tag{1}$$

where $\bar{x}$ and $\bar{\bar{x}}$ are arbitrary points of U_x.

B. We shall use the notion of the order of growth of a function only for continuous increasing functions satisfying the condition $\varphi(t) \geq t > 0$. The corresponding definition is given for this case in a form convenient for our purposes: the **order of growth of the function** $\varphi(t)$ is **not less than the order of growth of the function** $\psi(t)$ if there exists a constant $\alpha > 0$ such that $\varphi(t) \geq \alpha\psi(\alpha t)$.

The order of growth of the function $\varphi(t)$ is denoted by $[\varphi]$. If $[\varphi] \geq [\psi]$ and $[\psi] \geq [\varphi]$, then $[\varphi] = [\psi]$.

2. Let R be a metric space, and let the function $\varphi(t)$ satisfy the conditions formulated in 1.B. For each $\beta > 0$ construct in R the set ${}_{\varphi}P_{\beta}(R)$. By definition, $x \in {}_{\varphi}P_{\beta}$ if there exists a singular cycle Z, homologous to zero in R, such that $x \in |Z|$ and for every chain X, $\partial X = Z$, we have

$$d(X) > \beta\varphi d(Z). \tag{2}$$

(Roughly speaking, it is required that the diameter of the chain relative to the diameter of the cycle have order not lower than the order of growth of the function $\varphi(t)$; the condition $\varphi(t) \geq t$ is a natural consequence of the fact that $d(X) \geq d(Z)$.) It is clear that if $\beta_1 < \beta_2$, then $\beta_2\varphi(\beta_2 d(Z)) > \beta_1\varphi(\beta_1 d(Z))$, i.e.

$${}_{\varphi}P_{\beta_2} \subset {}_{\varphi}P_{\beta_1}. \tag{3}$$

Denote by $H_r({}_{\varphi}P_{\beta})$ the kernel of the natural homomorphism of the r-dimensional singular homology group of the set ${}_{\varphi}P_{\beta}$ into the r-dimensional singular homology group of the space R. By virtue of (3), for $\beta_1 < \beta_2$ there is a natural homomorphism

$$h = h_{\beta_2\beta_1} : H_r({}_{\varphi}P_{\beta_2}) \to H_r({}_{\varphi}P_{\beta_1}),$$

and for $\beta_1 < \beta_2 < \beta_3$ we have $h_{\beta_2\beta_1}h_{\beta_3\beta_2} = h_{\beta_3\beta_2}$, i.e., the system of groups $H_r({}_{\varphi}P_{\beta})$ and homomorphisms h forms an inverse spectrum $\{H_r({}_{\varphi}P_{\beta}), h\}$. We shall denote the limiting group of this spectrum by ${}_{\varphi}Q_r(R)$.

Theorem 1. *If $[\varphi] \geq [\psi]$, then there exists a canonical homomorphism of the group ${}_{\varphi}Q_r(R)$ into the group ${}_{\psi}Q_r(R)$.*

Let $x \in {}_{\varphi}P_{\beta}$; then there is a cycle Z from the definition of the set ${}_{\varphi}P_{\beta}$ such that, for any chain X, $\partial X = Z$, we have

$$d(X) > \beta\varphi(\beta d(Z)).$$

Since $[\varphi] \geq [\psi]$, there exists an $a > 0$ such that $\varphi(t) \geq a\psi(at)$. Hence

$$d(X) > \beta\varphi(\beta d(Z)) \geq a\beta\psi(a\beta d(Z)),$$

i.e.,

$${}_{\varphi}P_{\beta} \subset {}_{\psi}P_{a\beta}. \tag{4}$$

The identity mapping $R \to R$, by virtue of (4), gives rise to natural homomorphisms

$$f_{\beta} : H_r({}_{\varphi}P_{\beta}) \to H_r({}_{\psi}P_{a\beta}),$$

which together with the homomorphisms h form the commutative diagram

$$\begin{array}{ccc} H_r({}_\varphi P_{\beta_2}) & \xrightarrow{h} & H_r({}_\varphi P_{\beta_1}) \\ f_{\beta_2}\downarrow & & \downarrow f_{\beta_1} \\ H_r({}_\psi P_{a\beta_2}) & \xrightarrow{h} & H_r({}_\psi P_{\beta_1}) \end{array}$$

Therefore the system of homomorphisms f_β may be regarded as a homomorphism of the spectrum $\{H_r({}_\varphi P_\beta), h\}$ into the spectrum $\{H_r({}_\psi P_{a\beta}), h\}$. This homomorphism gives rise to a canonical homomorphism $f = f_{\varphi\psi}$ of the limiting groups of these spectra. It remains to note that the limiting groups of the spectra $\{H_r({}_\psi P_{a\beta}), h\}$ and $\{H_r({}_\psi P_\beta), h\}$ may be identified.

Remark. It is easy to see that from $[\varphi_1] \le [\varphi_2] \le [\varphi_3]$ it follows that

$$f_{\varphi_3\varphi_1} = f_{\varphi_2\varphi_1} f_{\varphi_3\varphi_2}.$$

Theorem 2. *If $[\varphi] = [\psi]$, then the groups ${}_\varphi Q_r(R)$ and ${}_\psi Q_r(R)$ are canonically isomorphic.*

By the preceding theorem we have canonical homomorphisms

$$f : {}_\varphi Q_r(R) \to {}_\psi Q_r(R)$$

and

$$g : {}_\psi Q_r(R) \to {}_\varphi Q_r(R).$$

It remains to verify that gf and fg are the identity mappings. It suffices to do this for gf. The mapping gf is obtained from a homomorphism of the inverse spectrum $\{H_r({}_\varphi P_\beta), h\}$ into the spectrum

$$\{H_r({}_\varphi P_{m\beta}), h\}$$

(m is some constant independent of β), generated by the inclusions

$${}_\varphi P_\beta \to {}_\psi P_{a\beta} \to {}_\varphi P_{m\beta}.$$

This homomorphism of spectra corresponds to the identity mapping of the group ${}_\varphi Q_r(R)$ into itself.

By Theorem 2, to each order of growth $[\varphi]$ there corresponds a group, which we denote by ${}_{[\varphi]}Q_r(R)$. It is easy to see that in Theorem 1, instead of the groups ${}_\varphi Q_r(R)$ and ${}_\psi Q_r(R)$, one may speak respectively of the groups ${}_{[\varphi]}Q_r(R)$ and ${}_{[\psi]}Q_r(R)$. The system of groups ${}_{[\varphi]}Q_r(R)$, together with the homomorphisms

$$f_{\varphi\psi} : {}_{[\varphi]}Q_r(R) \to {}_{[\psi]}Q_r(R),$$

forms, in the partially ordered directed set of orders of growth, an inverse spectrum

$$\{{}_{[\varphi]}Q_r(R), f\}.$$

We shall call it the spectrum of uniform homologies.

Example 1. Consider the surfaces Γ_1 and Γ_2, given respectively by the equations

$$z = x^2 + y^2$$

and

$$z = x^4 + y^4$$

in Euclidean space E^3, and the functions $\varphi_1(t) = t$, $\varphi_2(t) = t^2$, $\varphi_3(t) = t^4$. It can be shown that for Γ_1 the group ${}_{[\varphi_1]}Q_1$ is free cyclic, ${}_{[\varphi_2]}Q_1$, ${}_{[\varphi_3]}Q_1$ are trivial, while for Γ_2 the groups ${}_{[\varphi_1]}Q_1$, ${}_{[\varphi_2]}Q_1$ are free cyclic, and ${}_{[\varphi_3]}Q_1$ is trivial.

Theorem 3. *An equimorphism $g : R \to R'$ of spaces with uniform metric gives rise to an isomorphism*

$$g : {}_{[\varphi]}Q_r(R) \to {}_{[\varphi]}Q_r(R').$$

It is easy to see (cf. (4)) that if R is a space with a uniform metric, then there exists a $q > 0$ such that for $\beta > q$, in constructing the sets ${}_{\varphi}P_\beta$, one may restrict oneself to considering only those cycles for which

$$d(Z) \geq \gamma_2$$

(for the notation see 1.A). We note that γ_1 and γ_2 may be taken to be ob–

for both manifolds. Since g is an equimorphism, from $\rho(x, y) \geq \gamma_2$, $x, y \in R$, it follows that $\rho(g(x), g(y)) \geq l$, where l is a constant independent of the choice of the points $x, y \in R$. Let c denote the smaller of the numbers γ_2 and l. Then ([2]) there exist such positive constants C_1 and C_2 that

$$C_1 \leq \rho(x, y)/\rho(g(x), g(y)) \leq C_2 \tag{5}$$

as soon as $\rho(x, y) > c$ or $\rho(g(x), g(y)) > c$. Let now $x \in {}_{\varphi}P_\beta(R)$, $\beta > q$. This means that there is a cycle Z, $x \in |Z|$, $d(Z) \geq \gamma_2$, such that for any chain X, $\partial X = Z$, the inequality (2) holds. Denote by Z' and X' the images of the cycle Z and the chain X, respectively, under the mapping g. Then, as is easily obtained, $C_1 \leq d(Z)/d(Z') \leq C_2$ and $C_1 \leq d(X)/d(X') \leq C_2$; hence

$$d(X') \geq \frac{1}{C_2} d(X) \geq \frac{1}{C_2}\, \beta\varphi(\beta d(Z)) \geq \frac{1}{C_2}\, \beta\varphi(\beta C_1 d(Z')) \geq a\beta\varphi(a\beta d(Z')),\ a =$$

$$= \min\left(C_1, \frac{1}{C_2}\right).$$

Thus, $g(x) \in {}_{\varphi}P'$, i.e.

$$g({}_{\varphi}P_{\beta}) \subset {}_{\varphi}P'_{\alpha\beta} \tag{6}$$

(here ${}_{\varphi}P'_{\alpha\beta} = {}_{\varphi}P_{\alpha\beta}(R')$). Similarly, the existence of such a constant b is proved that for $\beta > q$ we shall have

$$g^{-1}({}_{\varphi}P'_{\beta}) \subset {}_{\varphi}P_{b\beta}. \tag{7}$$

Using the inclusions (6) and (7), by arguments analogous to those given in the proofs of Theorems 1 and 2, we obtain that the mapping g induces a canonical isomorphism.

Remark. In the case of geodesic spaces the groups ${}_{[\varphi]}Q_r$ are invariants of strong homeomorphisms.

Example 2. Returning to the surfaces Γ_1 and Γ_2 considered in Example 1, we obtain, by Theorem 3, that they are not equimorphic, although the groups $Q_r = {}_{\varphi_1}Q_r$ (see (4)) are isomorphic for them. Note, however, that the non-equimorphism of these surfaces can also be established by comparing their volume invariants ([1]). For the non-equimorphic manifolds considered in the following example, the volume invariants are the same and the groups Q_r are isomorphic.

Example 3. Consider the manifolds Π_1 and Π_2, given in Euclidean space E^4 by the equations $u = x^2 + y^2 - z^2$ and $x^4 + y^4 - z^2$, respectively. Let $\psi(t) = t^2$. It can be proved that ${}_{[\psi]}Q_1(\Pi_2)$ is a free cyclic group, while ${}_{[\psi]}Q_1(\Pi_1)$ is trivial. It follows that Π_1 and Π_2 are not equimorphic.

A strengthening of Theorem 3 is

Theorem 4. *An equimorphism $g : R \to R'$ of spaces with a uniform metric induces an isomorphism of spectra*

$$\hat{g} : \{[_{\varphi}]Q_r(R), f\} \to \{[_{\varphi}]Q_r(R'), f\}.$$

The assertion of the theorem follows from Theorem 3 and the commutativity of the diagrams

$$\begin{array}{ccc} {}_{[\varphi]}Q_r(R) & \to & {}_{[\psi]}Q_r(R) \\ \downarrow & & \downarrow \\ {}_{[\varphi]}Q_r(R') & \to & {}_{[\psi]}Q_r(R') \end{array} \qquad \begin{array}{ccc} {}_{[\varphi]}Q_r(R') & \to & {}_{[\psi]}Q_r(R') \\ \downarrow & & \downarrow \\ {}_{[\varphi]}Q_r(R) & \to & {}_{[\psi]}Q_r(R) \end{array}$$

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Note: Figure translations are in progress. See original paper for figures.