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# ISOMETRIC IMMERSIONS OF CLASS $C^1$

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## Abstract

## Full Text

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*MATHEMATICS*

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# ISOMETRIC IMMERSIONS OF CLASS $C^1$

*(Presented by Academician A. D. Aleksandrov, March 27, 1970)*

An immersion in Euclidean space  $\mathbf{R}^3$  of a two-dimensional Riemannian manifold by a surface  $F$  of class  $C^1$  or even  $C^{1,\alpha}$  for small  $\alpha$ , as follows from works <sup>(1-3)</sup>, may be such that for the surface  $F$  the relations between its intrinsic and extrinsic geometry that are characteristic of surfaces of class  $C^2$  are violated. Thus, for example, a closed surface with positive intrinsic curvature need not be convex, a surface with negative curvature need not be saddle-shaped, and a surface isometric to the plane need not be a cylinder.

In this connection it makes sense to consider isometric immersions in  $\mathbf{R}^3$  of class  $C^1$  subject to certain additional restrictions.

Such restrictions include the condition that the extrinsic curvature of the surface be bounded <sup>(4)</sup> and the condition that the surface belong to the class  $C^{1,\alpha}$  for  $\alpha > 2/3$  <sup>(5)</sup>.

In the present note, the condition of  $\mathfrak{A}$ -regularity is imposed on isometric immersions by a surface of class  $C^1$ .

**Definition.** An isometric immersion of a metric of class  $K$  by a surface  $F$  in  $\mathbf{R}^3$  is called **regular** in the class  $K$  with respect to a group of affine transformations  $\mathfrak{A}$  of the space  $\mathbf{R}^3$  ( $\mathfrak{A}$ -regular) if, under any affine transformation of  $\mathbf{R}^3$ , the surface  $F$  is transformed into a surface that is an isometric immersion of some metric of the same class  $K$ .

Immersion in  $\mathbf{R}^n$  regular with respect to the group of transformations of  $\mathbf{R}^n$  were considered in <sup>(6)</sup>.

We denote by  $A, A^+, A^-, A^0$ , respectively, the classes of metrics of bounded curvature, nonnegative, nonpositive curvature, and zero curvature (the class of locally Euclidean metrics).

Theorems 1-3 establish structural properties of  $\mathfrak{A}$ -regular immersions of metrics of the classes  $A^+, A^-, A^0$ . Theorems 4-6 establish connections between the conditions of bounded extrinsic curvature,  $\mathfrak{A}$ -regularity, and membership of the surface in the class  $C^{1,\alpha}$  for  $\alpha > 2/3$ . At the same time one should additionally note that the immersion of a metric of any of the classes  $A, A^+, A^-, A^0$  by a

surface of bounded extrinsic curvature is, obviously,  $\mathfrak{A}$ -regular. Theorem 6 is a generalization of an analogous theorem of Yu. F. Borisov <sup>(5)</sup>.

**Theorem 1.** *If a surface  $F \in C^1$  is an  $\mathfrak{A}$ -regular immersion in  $\mathbf{R}^3$  of a metric of class  $A^+$ , then for every point  $M$  of it at least one of the following assertions is true:*

( $\alpha$ ). *Some neighborhood of the point  $M$  on the surface  $F$  is a convex surface (in particular, this neighborhood may be a plane domain).*

( $\beta$ ). *Through the point  $M$  there passes a rectilinear segment  $\pi(M)$  lying on the surface  $F$ , its ends are situated on the boundary of  $F$ , and the tangent plane to  $F$  along the segment  $\pi(M)$  is stationary. If the point  $M$  has no plane neighborhood on the surface  $F$ , then such a segment is unique and none of its points has a plane neighborhood on  $F$ .*

**Theorem 1'.** *If a surface  $F$  satisfies the conditions of Theorem 1, then every cap cut from the surface  $F$  by a plane is a convex surface.*

**Theorem 2.** *If a surface  $F$  is an  $\mathfrak{A}$ -regular immersion in  $\mathbf{R}^3$  of a metric of class  $A^-$ , then  $F$  is a saddle surface.*

**Theorem 3.** *If a surface  $F$  in  $\mathbf{R}^3$  of class  $C^1$  is an  $\mathfrak{A}$ -regular immersion of a metric of class  $A^0$ , then for any of its points  $M$  at least one of the following assertions is true:*

( $\alpha$ ). *Some neighborhood of the point  $M$  on the surface  $F$  is a planar domain.*

( $\beta$ ). *Through the point  $M$  there passes a rectilinear segment  $\pi(M)$ , lying on the surface  $F$ , with endpoints on the boundary of  $F$ ; the tangent plane to  $F$  along  $\pi(M)$  is stationary. If the point  $M$  has no planar neighborhood on the surface  $F$ , then such a segment is unique and none of its points has a planar neighborhood on  $F$ .*

**Theorem 4.** *If a surface  $F$  of class  $C^{1,\alpha}$  ( $\alpha > 1/2$ ) is an  $\mathfrak{A}$ -regular immersion in  $\mathbf{R}^3$  of a metric of one of the classes  $A^+$ ,  $A^-$ ,  $A^0$ , then  $F$  is a surface of bounded extrinsic curvature.*

**Theorem 5.** *If a surface  $F$  of class  $C^{1,\alpha}$  ( $\alpha > 2/3$ ) is an immersion in  $\mathbf{R}^3$  of a metric of one of the classes  $A$ ,  $A^+$ ,  $A^-$ ,  $A^0$ , then this immersion  $F$  is an  $\mathfrak{A}$ -regular immersion.*

From Theorems 1-5 it follows that

**Theorem 6.** *If a surface  $F \in C^{1,\alpha}$  ( $\alpha > 2/3$ ) is an immersion in  $\mathbf{R}^3$  of a metric of one of the classes  $A^+$ ,  $A^-$ ,  $A^0$ , then it is a surface of bounded extrinsic curvature and for it the assertion of the corresponding structural Theorem 1, 2, or 3 is satisfied.*

The **proof** of Theorem 1 is based on the following lemmas.

**Lemma 1.** *If a surface  $F$  satisfies the conditions of Theorem 1, is given by the equation  $z = f(x, y)$ ,  $(x, y) \in K$ , where  $K$  is a closed domain in the  $x, y$ -plane,*

and if at the interior points of  $K$   $f(x, y) > 0$ , while on the boundary of  $K$   $f(x, y) = 0$ , then  $F$  is a convex surface.

The central point in the proof of Lemma 1 is the following assertion: some neighborhood of the “vertex” of any cap cut by a plane from the surface  $F$  is a convex surface.

**Lemma 2.** If a surface  $F \in C^1$  is saddle and is an isometric immersion of a metric of class  $A^0$ , then for it the assertions of Theorem 2 are fulfilled.

**Proof.** Introduce on the surface  $F$  a new topology induced by the Gauss map of  $F$  onto the sphere. Divide  $F$  into two sets  $F^0$  and  $F' = F \setminus F^0$ , where  $F^0$  is the set of points of  $F$  interior with respect to the topology introduced above. If  $F^0$  is nonempty, then the extrinsic curvature of  $F^0$  coincides with the intrinsic curvature, which is proved by approximating the surface  $F^0$  by a  $C^1$ -sequence of saddle polyhedra. Hence, by a theorem of A. V. Pogorelov <sup>(4)</sup>, it follows that for the points of  $F^0$  the assertions of Theorem 2 are fulfilled.

The proof of Theorem 3 is given in <sup>(6)</sup>.

The proof of Theorem 4 is carried out analogously to the proof of Lemma 2. We note only that from the condition  $F \in C^{1,\alpha}$  ( $\alpha > 1/2$ ) it follows that the image of the set  $F'$  on the sphere has measure zero.

The proof of Theorem 5 is based on the work <sup>(7)</sup>. Let  $F \in C^{1,\alpha}$  ( $\alpha > 2/3$ ), and let the intrinsic metric of  $F$  belong to class  $A$ . Define on  $F$  a set function  $\sigma$ , completely additive on the ring of Borel sets, which on each domain  $G \subset F$  with rectifiable boundary  $L$  coincides with the quantity  $\Delta(L')$ , where  $L'$  is the spherical image of  $L$ , and  $\Delta$  is the oriented area of  $L'$ .

Let  $\alpha$  be an affine transformation of  $\mathbb{R}^3$ , and let  $G$  be a domain on  $F$  bounded by a smooth curve.

If the variation  $|\sigma(F)| < \infty$ , then the formula holds

$$\sigma(\alpha(G)) = \int_G J(x') \sigma(dG),$$

where  $x'$  is the spherical image of the point  $x \in G$ , and  $J(x')$  is the Jacobian of the diffeomorphism  $\beta$  of the unit sphere  $S$  onto itself, defined by the equality  $x' = \beta(\alpha(x'))$ , for all  $x' \in S$ . It follows from <sup>(7)</sup> that in order for the surface  $F \in C^1$ , ( $\alpha > 1/2$ ), to be an isometric immersion of a metric of class  $A$ , it is necessary and sufficient that the variation  $|\sigma(F)| < \infty$ . In this case the intrinsic curvature  $\omega(F) \equiv \sigma(F)$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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