ON SPATIALLY HOMOGENEOUS GRAVITATIONAL FIELDS

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Abstract

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We shall call a metric spatially homogeneous (s.h.) if it admits a simply or multiply transitive group of motions $\breve{G}_r$ ($r \geq 3$), acting on spacelike hypersurfaces. Such a definition is often encountered in the literature ($^{1-3}$), although it is not entirely justified from the physical point of view (^{3}). Let us consider the case $r = 3$. Metrics satisfying the stated requirements are known. Usually one chooses a semigeodesic (synchronous) coordinate system in which the hypersurfaces of transitivity have equations $x^0 =$ const, and each component of the Killing vectors ξ^i is at most a function of x^1, x^2, x^3. Then the metrics obtained as a result of integrating the Killing equations can be written in the form (^{1,2})*

$$ds^2 = dx^{0\,2} + g_{ik}\,dx^i dx^k = c^2 dt^2 + \alpha_{ab} e^a_i e^b_k dx^i dx^k.$$

Here α_{ab} are arbitrary functions of t, and e^a_i are Killing vectors of the group $\breve{G}_3$, reciprocal to the given ones (^{4}). The quantities e^i_a are defined by the relations $e^a_i e^k_a = \delta^k_i$, $e^a_i e^i_b = \delta^a_b$. For each of the 9 types of real nonisomorphic structures (^{5,7}), e^a_i can be chosen so that the structure constants $\bar{C}^a_{bc}$ from the relations

$$e^a_{i,k} - e^a_{k,i} = \bar{C}^a_{bc} e^b_i e^c_k \tag{1}$$

coincide with the structure constants C^a_{bc} of the given G_3 ($\xi^i_{b,k}\xi^k_c - \xi^i_{c,k}\xi^k_b = C^a_{bc}\xi^i_a$). The form of the operators ξ^i_a and e^a_i is given in the Appendix.

S.h. metrics satisfying the Einstein equations $R_{\mu\nu} = -\varkappa(T_{\mu\nu} - {}^1/{}_2 g_{\mu\nu} T)$ are being studied intensively in connection with their cosmological applications. As the energy-momentum tensor of the medium one most often takes

$$T_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu - p g_{\mu\nu}, \tag{2}$$

where ρc^2 and p are the energy density and pressure; u_μ are the components of the velocity of the medium, which in the present case have the form (^{2})

$u_0 = n_0(t)$, $u_i = n_a(t)e^a_i$. Generally speaking, u^i are not equal to zero, i.e. the synchronous coordinate system used is not comoving with the medium. However, in cosmological problems we are primarily interested in the behavior of the medium—its density, deformation, rotation, etc.; for this it is convenient to use a comoving reference system (c.r.s.). And in those cases where one seeks solutions satisfying previously imposed requirements (for example, the requirement of a specified dependence of the density on the proper time τ of an element of the medium—which is important in calculating the chemical composition of matter), it is practically impossible to do without an s.s.r. The aim of the article is to derive the Einstein equations in an s.s.r. for all types of spatially homogeneous metrics.

There exist coordinate transformations (containing 4 arbitrary functions of t) that leave invariant the form of ξ^a_i and the equation $x^0 = \text{const}$. Having found these transformations, it is easy to show that, by a choice of the arbitrary functions, all u^i can be made equal to zero, i.e., an s.s.r. can be introduced. The synchronicity of the coordinate system is thereby violated. Suppose that the choice of s.s.r. has been made, and integrate Killing' s equations. As a result we obtain

$$g_{00} = \alpha_{00}, \qquad g_{0i} = \alpha_{0a}e^a_i, \qquad g_{ik} = \alpha_{ab}e^a_i e^b_k, \tag{3}$$

where the α' s are arbitrary functions of t. In view of the possibility of direct physical interpretation of the quantities entering the Einstein equations, we write these equations in the chronometrically invariant (ch.i.) form proposed by Zelmanov (^{6}):

$$^*\dot{D} + D_{ik}D^{ik} + A_{ik}A^{ki} + {}^*\nabla^i_{iF} - F^i_{iF} = -\frac{1}{2}(\rho + U), \tag{4}$$

$$^*\dot{D}_{ik} - (D_{ij} + A_{ij})(D^j_k + A^j_k) + DD_{ik} - D_{ij}D^j_k + 3A_{ij}A^j_k +$$

$$+\frac{1}{2}({}^*\nabla_{iF}k + {}^*\nabla_{kF}i) - F_{iF}k - C_{ik} = \frac{1}{2}(\rho h_{ik} + 2U_{ik} - Uh_{ik}),$$

$$^*\nabla_k(h^{ik}D - D^{ik} - A^{ik}) + 2F^{ik}_{kA} = J^i,$$

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[*] Greek indices $\mu, \nu, ...$ take the values 0, 1, 2, 3; Latin $i, j, k, ...$, 1, 2, 3. "Frame" indices $a, b, c, d, ...$ take the values 1, 2, 3. A comma with an index denotes partial differentiation.

The constants c and $\varkappa$ are equal to 1; a dot denotes $\frac{\partial}{\partial t}$, while a dot with an asterisk denotes ${}^*\frac{\partial}{\partial t}$. Ch.i. quantities and operators (marked by $*$) are expressed in terms of the ordinary operators and the metric $g_{\mu\nu}$ as follows:

$$ {}^*\frac{\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}; \qquad {}^*\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial t}; $$

the metric tensor

$$ h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}; $$

the tensor of deformation velocities

$$ D_{ik} = \frac{1}{2}{}^*\dot{h}_{ik}; $$

the vector of gravitational-inertial force

$$ F_i = (1-w)^{-1}(w_{,i} - \dot{V}_i); $$

the tensor of angular velocity of rotation

$$ A_{ik} = \frac{1}{2}(V_{k,i} - V_{i,k}) + \frac{1}{2}(F_{iV}k - F_{kV}i), $$

where the auxiliary quantities w, V_i are defined by $g_{00} = (1-w)^2,\ g_{0i} = -V_i(1-w)$. Further, the Ricci tensor of space is

$$ C_{lk} = H_{lk} - A_{ki}D^i_l - A_{li}D^i_k - A_{kl}D, \quad \text{where } H_{lk} = H_{lki}{}^i; $$

$$ H_{lki}{}^j Q_j = ({}^*\nabla_{ik} - {}^*\nabla_{ki})Q_l - 2A_{ik}{}^*\dot{Q}_l; \tag{5} $$

Q_l is an arbitrary vector; ${}^*\nabla_i$ is the operator of covariant differentiation, constructed according to the usual rules from the metric h_{ik} and ${}^*\frac{\partial}{\partial x^i}$. Finally,

$$ \rho = T_{00}(g_{00})^{-1}, \qquad J^i = T^i_0(g_{00})^{-1/2}, \qquad U^{ik} = T^{ik}. $$

For the tensor (2), and in an s.s.r.,

$$ J^i = 0, \qquad U^{ik} = ph^{ik}. $$

For spatially homogeneous metrics (3), all the vector and tensor quantities considered can be "expanded" with respect to the frame vectors e_i^a, with coefficients depending only on t. For example,

$$h_{ik} = \left(-\alpha_{ab} + \frac{\alpha_{0a}\alpha_{0b}}{\alpha_{00}}\right) e_i^a e_k^b = \gamma_{ab}(t) e_i^a e_k^b.$$

With the help of γ_{ab}, frame indices are shifted. We denote the frame components of the quantities $D_{ik}, V_i, F_i, A_{ik}, C_{lk}$ by the corresponding lowercase letters; then

$$d_{ab} = \frac{1}{2} {}^*\dot{\gamma}_{ab}, \qquad v_a = -\frac{\alpha_{0a}}{\sqrt{\alpha_{00}}}, \qquad f_a = -{}^*\dot{v}_a,$$

and, taking (1) into account,

$$a_{ab} = \frac{1}{2} v_c C^c{}_{ba} + \frac{1}{2}(f_{av} b - f_{bv} a).$$

We find how c_{ab} is expressed in terms of $\gamma_{ab}, v_a, C^a{}_{bc}$. In (5), as Q_i we substitute e_i^d. We introduce the notation $-\Gamma^d{}_{ac} =$

$$= e_a^i e_c^{k*} \nabla_k e_i^d.$$

Then

$$e_a^i e_b^j e_c^{k*} \nabla_j e_{ki}^d = -\Gamma^d_{ac\oplus b} + \Gamma^d_{gc} \Gamma^g_{ab} + \Gamma^d_{ag} \Gamma^g_{cb},$$

where

$$\Gamma^d_{ac\oplus b} = e_b^{j*} \nabla_j \Gamma^d_{ac}.$$

Since

$${}^*\left(e_l^d\right)^{\bullet} = 0,$$

from (5) we obtain

$$H_{ab} = H_{abc}{}^c = -2\Gamma^c_{a[c\oplus b]} + 2\Gamma^c_{d[b} \Gamma^d_{a|c]} + 2\Gamma^c_{ad} \Gamma^d_{[bc]}, \tag{6}$$

where $X_{[a|b|c]} = \frac{1}{2}(X_{abc} - X_{cba})$. The required c_{ab} are related to H_{ab} by

$$c_{ab} = H_{ab} - a_{bc} d_a{}^c - a_{ac} d_b{}^c - a_{ba} d. \tag{7}$$

Let us compute the quantities Γ^c_{ab}. Applying the operator ${}^*\nabla_k$ to $\gamma_{ab} = e_a^i e_{bi}$, we get

$$\Gamma_{abc} + \Gamma_{bac} = v_c^* \gamma_{ab},$$

and from the definition of Γ^d_{ac} and (1) it follows that

$$\Gamma^d_{bc} - \Gamma^d_{cb} = C^d{}_{cb}.$$

If one uses the equalities obtained from these by a cyclic interchange of indices, one can find

$$-\Gamma_{abc} = \frac{1}{2}(C_{abc} + C_{bca} - C_{cab} - v^*_c\gamma_{ab} - v^*_b\gamma_{ac} + v^*_a\gamma_{bc}).$$

After minor transformations,

$$-\Gamma^c_{ab} = \frac{1}{2}\left[C^c{}_{ab} + \gamma^{cd}(\gamma_{ag}C^g{}_{bd} + \gamma_{bg}C^g{}_{ad})\right] - v_a d_b{}^c - v_b d_a{}^c + v^c d_{ab}. \tag{8}$$

Since Γ^c_{ab} depend only on t, we have

$$\Gamma^c_{ab\oplus d} = v^*_d(\Gamma^c_{ab})^\bullet,$$

and, taking this fact into account, formulas (6)–(8) solve the problem posed. In the Appendix are given $\bar{c}_{ab}$, computed by (6)–(8) under the condition $a_{00} = 1$, $a_{0a} = 0$. Let us also introduce the notation

$$^*\nabla_i = e^c_i\Box_c.$$

It is not difficult to prove that for any quantity

$$X^{\dots k}_{i\dots} = x^{\dots b}_{a\dots}(t)e^a_i \cdots e^k_b$$

one has

$$^*\nabla_j X^{\dots k}_{i\dots} = e^c_j e^a_i \cdots e^k_b\,\Box_c x^{\dots b}_{a\dots},$$

where

$$\Box_c x^{\dots b}_{a\dots} = v^*_c\left(x^{\dots b}_{a\dots}\right)^\bullet - x^{\dots b}_{d\dots}\Gamma^d_{ac} - \dots + x^{\dots d}_{a\dots}\Gamma^b_{dc}.$$

Finally, let us pass to proper time by means of

$$d\tau = \sqrt{a_{00}}\,dt$$

(which is equivalent to the choice $a_{00} = 1$) and write equations (4) in the s.s.o. in the form of a system of ordinary differential equations for functions of τ ($d/d\tau$ is denoted by a dot):

$$\begin{gathered}
\dot{d} + d_{ab}d^{ba} + a_{ab}a^{ba} + \Box_a f^a - f_a f^a = -\frac{1}{2}(\rho + 3p), \\
\Box_b(\gamma^{ab}d - d^{ab} - a^{ab}) + 2f_b a^{ab} = 0, \\
\dot{d}_{ab} - (d_{ac} + a_{ac})(d_b{}^c + a_b{}^c) + d d_{ab} - d_{ac}d_b{}^c + 3a_{ac}a_b{}^c \\
+\frac{1}{2}(\Box_a f_b + \Box_b f_a) - f_a f_b - c_{ab} = \frac{1}{2}(\rho - p)\gamma_{ab}.
\end{gathered} \tag{9}$$

Let us note that from the hydrodynamic equations

$$\dot{\rho} + d(\rho + p) = 0, \qquad v_a\dot{p} = (\rho + p)f_a,$$

which are a consequence of (9), it follows that

$$f_a v_b - f_b v_a = 0,$$

i.e.

$$a_{ab} = \frac{1}{2} v_c C^c{}_{ba}.$$

Moreover,

$$v_a = -a_{0a} = k_a \exp \int \frac{dp}{\rho + p},$$

where k_a are arbitrary constants.

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Appendix

Introduce the notation

$$X_a = \xi_a^i \frac{\partial}{\partial x^i} = \xi_a^i p_i;$$

denote the determinant of the matrix γ_{ab} by γ; then:

Type I.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3, \quad e_i^1 = (1,0,0), \quad e_i^2 = (0,1,0), \quad e_i^3 = (0,0,1), \quad \bar{c}_{ab} = 0.$$

Type II.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = x^2 p_1 - p_3, \quad e_i^1 = (1, x^3, 0), \quad e_i^2 = (0,1,0),$$

$$e_i^3 = (0,0,1), \quad \bar{c}_1^2 = \bar{c}_1^3 = \bar{c}_2^3 = 0, \quad -\bar{c}_1^1 = \bar{c}_2^2 = \bar{c}_3^3 = \frac{1}{2\gamma}\gamma_{11}^2.$$

Type III.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = x^1 p_1 - p_3, \quad e_i^1 = (e^{x^3}, 0, 0), \quad e_i^2 = (0,1,0),$$

$$e_i^3 = (0,0,1), \quad \bar{c}_1^3 = \bar{c}_2^3 = 0, \quad \bar{c}_1^2 = -\frac{1}{\gamma}\gamma_{11}\gamma_{12}, \quad \bar{c}_1^1 = \bar{c}_3^3 = \gamma^{33} + \frac{1}{2\gamma}\gamma_{12}^2, \quad \bar{c}_2^2 = -\frac{1}{2\gamma}\gamma_{12}^2.$$

Type IV. $X_1 = p_1$, $X_2 = p_2$, $X_3 = (x^1 + x^2)p_1 + x^2 p_2 - p_3$, $e_i^1 = (e^{x^3}, x^3 e^{x^3}, 0)$, $e_i^2 = (0, e^{x^3}, 0)$, $e_i^3 = (0,0,1)$, $\bar{c}_1^3 = \bar{c}_2^3 = 0$, $\bar{c}_1^2 = \frac{1}{\gamma}\gamma_{11}^2$, $\bar{c}_1^1 = 2\gamma^{33} - \frac{1}{2\gamma} \times \gamma_{11}(\gamma_{11} + 2\gamma_{12})$, $\bar{c}_2^2 = 2\gamma^{33} + \frac{1}{2\gamma}\gamma_{11}(\gamma_{11} + 2\gamma_{12})$, $\bar{c}_3^3 = 2\gamma^{33} + \frac{1}{2\gamma}\gamma_{11}^2$.

Type V. $X_1 = p_1$, $X_2 = p_2$, $X_3 = x^1 p_1 + x^2 p_2 - p_3$, $e_i^1 = (e^{x^3}, 0, 0)$, $e_i^2 = (0, e^{x^3}, 0)$, $e_i^3 = (0,0,1)$, $\bar{c}_{ab} = 2\gamma^{33}\gamma_{ab}$.

Type VI. $X_1 = p_1,\ X_2 = p_2,\ X_3 = x^1p_1 + qx^2p_2 - p_3,\ e_i^1 = (e^{x^3}, 0, 0),\ e_i^2 = (0, e^{qx^3}, 0),\ e_i^3 = (0, 0, 1),\ \bar{c}_1^3 = \bar{c}_2^3 = 0,\ \bar{c}_1^2 = \frac{1}{\gamma}\gamma_{11}\gamma_{12}(q-1),\ \bar{c}_1^1 = \frac{1}{2\gamma}\gamma_{12}^2 \times (1 - q^2) + (1+q)\gamma^{33},\ \bar{c}_2^2 = -\frac{1}{2\gamma}\gamma_{12}^2(1-q)^2 + q(1+q)\gamma^{33},\ \bar{c}_3^3 = \frac{1}{2\gamma}\gamma_{12}^2(1-q)^2 + (1 + q^2)\gamma^{33}$, where $q \neq 0, 1$.

Type VII. $X_1 = p_1,\ X_2 = p_2,\ X_3 = -x^2p_1 + (qx^2 + x^1)p_2 + p_3,\ e_i^1 = [e^{-\frac{1}{2}qx^3}(q\sin\frac{p}{2}x^3 + p\cos\frac{p}{2}x^3),\ 2e^{-\frac{1}{2}qx^3}\sin\frac{p}{2}x^3,\ 0],\ e_i^2 = [-2e^{-\frac{1}{2}qx^3} \times \sin\frac{p}{2}x^3,\ e^{-\frac{1}{2}qx^3}(-q\sin\frac{p}{2}x^3 + p\cos\frac{p}{2}x^3),\ 0],\ e_i^3 = (0, 0, 1),\ q^2 < 4,\ p = \sqrt{4 - q^2},\ \bar{c}_1^1 = \frac{1}{2\gamma}[\gamma_{22}^2 - (\gamma_{11} - q\gamma_{12})^2],\ \bar{c}_2^2 = -\frac{1}{2\gamma}[\gamma_{22}^2 + (\gamma_{11} + q\gamma_{12})^2 - 2\gamma_{11}(\gamma_{11} + q\gamma_{22})],\ \bar{c}_3^3 = \frac{1}{2\gamma}[-(\gamma_{11} + \gamma_{22} + q\gamma_{12})^2 + 2(\gamma_{11}^2 + \gamma_{22}^2 + 2\gamma_{12}^2) + 2q^2\gamma_{11}\gamma_{22}],\ \bar{c}_1^3 = \bar{c}_2^3 = 0,\ \bar{c}_1^2 = -\frac{1}{\gamma}[\gamma_{12}(\gamma_{11} + \gamma_{22}) - q\gamma_{11}\gamma_{22}]$.

Type VIII. $X_1 = p_2,\ X_2 = x^2p_2 + p_3,\ X_3 = e^{x^3}p_1 + x^{2\,2}p_2 + 2x^2p_3,\ e_i^1 = (1, x^1e^{-x^3}, -x^1),\ e_i^2 = (0, -2x^1e^{-x^3}, 1),\ e_i^3 = (0, e^{-x^3}, 0),\ \bar{c}_1^2 = -2\bar{c}_2^3 = -4\left(\gamma^{23} + \frac{1}{\gamma}\gamma_{12}A\right),\ \bar{c}_1^3 = -2\left(2\gamma^{33} - \frac{1}{\gamma}\gamma_{11}A\right),\ \bar{c}_1^1 = \bar{c}_3^3 = -2\left(2\gamma^{13} - \frac{1}{\gamma}\gamma_{22}A\right),\ \bar{c}_2^2 = 2\left[\gamma^{22} - \frac{1}{\gamma}(\gamma_{22} + \gamma_{13})A\right]$, where $A = \gamma_{22} - \gamma_{13}$.

Type IX. $X_1 = p_2,\ X_2 = \cos x^2p_1 - \operatorname{ctg} x^1 \sin x^2p_2 + \frac{\sin x^2}{\sin x^1}p_3,\ X_3 = -\sin x^2p_1 - \operatorname{ctg} x^1 \cos x^2p_2 + \frac{\cos x^2}{\sin x^1}p_3,\ e_i^1 = (\cos x^3, \sin x^3 \sin x^1, 0),\ e_i^2 = (-\sin x^3, \cos x^3 \sin x^1, 0),\ e_i^3 = (0, \cos x^1, 1),\ \bar{c}_1^2 = -2\gamma^{12} - \frac{1}{\gamma}\gamma_{12}A,\ \bar{c}_1^1 = \frac{1}{2\gamma}A^2 - B - \gamma^{11} - \frac{1}{\gamma}(\gamma_{11}^2 + \gamma_{12}^2 + \gamma_{13}^2)$, where $A = \gamma_{11} + \gamma_{22} + \gamma_{33},\ B = \gamma^{11} + \gamma^{22} + \gamma^{33}$; the remaining components $\bar{c}_a^b$ are obtained by a cyclic replacement of the indices $1, 2, 3$.

For some geometrical properties of the metrics considered, see (3,8).

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Note: Figure translations are in progress. See original paper for figures.