ON THE REPRESENTATION OF ENTIRE FUNCTIONS OF ARBITRARY ORDER OF GROWTH

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.535.4

MATHEMATICS

G. V. EFIMOV

ON THE REPRESENTATION OF ENTIRE FUNCTIONS OF ARBITRARY ORDER OF GROWTH

(Presented by Academician N. N. Bogolyubov, 24 IV 1970)

We shall prove a Fourier-type representation for entire functions of arbitrary order of growth. The proof carried out is a generalization of the method given by I. M. Gel' fand and G. E. Shilov ([1]) for entire functions of first order of growth. Another proof of the theorem on the representation of entire functions of arbitrary order of growth on the basis of the theory of Dirichlet series was given by A. F. Leont' ev ([2]).

We shall consider classes of entire analytic functions of one complex variable $z = x + iy$. Let $g(z)$ be an entire function; then

$$M_g(r) = \max_{|z|=r} |g(z)|.$$

Let $h(r)$ be a monotone twice differentiable function, and

$$h(r) > 0, \qquad h'(r) > 0, \qquad h''(r) > 0.$$

If

$$\lim_{r\to\infty} \frac{\ln M_g(r)}{h(r)} = 1,$$

then we shall say that the entire function $g(z)$ has exact order of growth $h(r)$. If $h(r) = ar^\rho$, then it is customary to say that the entire function has order of growth ρ and finite type a.

We shall say that the entire function $g(z)$ belongs to the space $\mathfrak{Z}_h$ if for this function $g(z)$ there exist constants $\delta > 0$ and $B > 0$ such that

$$|g(z)| \leq Be^{h((1-\delta)|z|)}. \tag{1}$$

We shall prove

Theorem. If $g(z) \in \mathfrak{Z}_h$, then this function can be represented in the form

$$g(z) = \int e^{iz\xi}\, d\sigma_g(\xi), \tag{2}$$

where $\sigma_g(\xi)$ is a complex completely additive measure in the complex plane ξ, and for this measure there always exists an $\varepsilon > 0$ such that the integral

$$\int e^{H((1+\varepsilon)|\xi|)}\, |d\sigma_g(\xi)| < \infty, \tag{3}$$

converges, where the function $H(s)$ is conjugate in the sense of Young (see, for example, ([3])) to the function $h(r)$, i.e.

$$H(s) = \max_{r>0}(sr - h(r)). \tag{4}$$

First of all let us prove that if the entire function $g(z)$ is representable in the form (2) with condition (3), then it belongs to $\mathfrak{Z}_h$. Indeed, $g(z)$ is entire, since the integral (2) converges for all z, and

$$|g(z)| \leq \int e^{|z|\cdot|\xi|} |d\sigma_g(\xi)| \leq \max_{|\xi|} e^{|z|\cdot|\xi| - H((1+\varepsilon)|\xi|)} \int e^{H((1+\varepsilon)|\xi|)} |d\sigma_g(\xi)| \leq$$

$$\leq B_1 \exp\left\{h\left(\frac{|z|}{1+\varepsilon}\right)\right\} = B_1 \exp\{h((1-\delta_1)|z|)\},$$

where

$$B_1 = \int |d\sigma_g(\zeta)| \exp\{H'((1+\varepsilon)|\zeta|)\} \quad \text{and} \quad \delta_1 = \frac{\varepsilon}{1+\varepsilon}.$$

Consequently, $g(z) \in \mathfrak{B}_h$.

Introduce the linear countably normed space of entire functions Z_H, for which the following system of norms is defined:

$$\|f\|_p = \sup_{\zeta} |f(\zeta)| \exp\{-H((1+1/p)|\zeta|)\} \tag{5}$$

for $p = 1, 2, 3, \ldots$. The functions $M_p(s) = \exp\{-H((1+1/p)s)\}$ satisfy the inequalities $0 < M_1(s) \leq M_2(s) \leq \cdots$ and the so-called condition (P):

For a given $\varepsilon > 0$ and any p, one can indicate a $p' > p$ and an N such that, for those s for which $s > N$, the inequality

$$M_p(s) < \varepsilon M_{p'}(s)$$

holds.

Under these conditions the norms (5) are consistent, so that the space Z_H is complete and, moreover, it is perfect ([1]).

If $f(\zeta) \in Z_H$, then for every $\varepsilon > 0$ there exists a number $C_\varepsilon > 0$ such that

$$|f(\zeta)| \leq C_\varepsilon e^{H((1+\varepsilon)|\zeta|)}. \tag{6}$$

Let us find the general form of a linear continuous functional on the space Z_H. It is enough to find a general linear functional (F, f) on the normed space $Z_H^{(p)}$ –the completion of the space Z_H with respect to the norm $\|f\|_p$. The space $Z_H^{(p)}$ consists of certain analytic functions $f(\zeta)$. This space is closed with respect to uniform convergence. Extending, by the Hahn-Banach theorem, the functional F to the space of all continuous functions and applying the Riesz-Radon theorem, we obtain

$$(F, f) = \int f(\zeta)\, d\mu(\zeta), \tag{7}$$

where $\mu(\zeta)$ is a complex completely additive measure in the complex plane ζ, for which the integral

$$\int |d\mu(\zeta)| \exp\left\{H\left(\left(1+\frac{1}{p}\right)|\zeta|\right)\right\}$$

is finite.

By virtue of theorem ([1]) on the structure of the space conjugate to a countably normed space, formula (7), for all possible p, gives the general form of a linear continuous functional on the space Z_H.

Next, the Taylor series

$$f(\zeta) = \sum_{n=0}^{\infty} f_n \zeta^n$$

converges in the topology of the space Z_H. Indeed, applying Cauchy' s formula and the estimate (6), we obtain

$$|f_n| = \left|\frac{1}{2\pi i}\oint \frac{d\zeta\, f(\zeta)}{\zeta^{n+1}}\right| \leq C_\varepsilon \min_{s>0} \frac{e^{H((1+\varepsilon)s)}}{s^n} = C_\varepsilon (1+\varepsilon)^n e^{-B(n)},$$

where

$$B(n) = \max_{s>0}(n \ln s - H(s)). \tag{8}$$

The norm $\|\zeta^n\|_p$ is, according to definition (5),

$$\|\zeta^n\|_p = \max_{s>0} s^n e^{-H((1+1/p)s)} = (1+1/p)^{-n} e^{B(n)}.$$

Then we have the estimate

$$\sum_{n=0}^{\infty} |f_n| \cdot \|\zeta^n\|_p \leq C_\varepsilon \sum_{n=0}^{\infty} \left[\frac{1+\varepsilon}{1+1/p}\right]^n. \tag{9}$$

Since the number ε can be chosen arbitrarily small, the series (9) converges for each given p.

Consequently,

$$(F, f) = \sum_{n=0}^{\infty} f_n F_n, \tag{10}$$

where $F_n = (F, \xi^n)$ is a fixed sequence of constants. Conversely, any sequence of constants F_n such that the series (10) converges for every function $f(\xi) \in Z_H$ and defines a continuous linear functional on the space Z_H by formula (10) can be represented in the form

$$F_n = \int \xi^n \, d\sigma(\xi), \tag{11}$$

which is obtained from the general formula (7) for $f(\xi) = \xi^n$.

Let now

$$g(z) = \sum_{n=0}^{\infty} g_n z^n$$

be an entire function from the space $\mathfrak{Z}_h$, i.e., satisfying an inequality of the form (1). We shall show that the numbers $F_n = (-i)^n g_n n!$ define a linear functional on the space Z_H, where the function $H(s)$ is conjugate in the sense of Young to the function $h(r)$ according to (4). Indeed, we have the following estimates:

$$|g_n| = \left| \frac{1}{2\pi i} \oint \frac{d_z g(z)}{z^n} \right| \leq B \min_{r>0} \frac{e^{h((1-\delta)r)}}{r^n} = B(1-\delta)^n e^{-A(n)},$$

where

$$A(n) = \max_{r>0}(n \ln r - h(r)), \tag{12}$$

and, according to the well-known Stirling formula,

$$n! = \sqrt{2\pi n}\, E_n e^{n \ln n - n},$$

where $E_n \to 1$ as $n \to \infty$. Further, we have

$$\sum_{n=0}^{\infty} |f_n F_n| \leq$$

$$\leq \sqrt{2\pi}\, C_\varepsilon B \sum_{n=0}^{\infty} \sqrt{n}\, E_n\, (1+\varepsilon)^n (1-\delta)^n \exp\{-B(n) - A(n) + n \ln n - n\}.$$

We shall show that

$$B(n) + A(n) \equiv n \ln n - n. \tag{13}$$

Indeed, according to (12), we have

$$A(n) = n \ln r(n) - h(r(n)), \tag{14}$$

where the function $r = r(n)$ is the solution of the following equation:

$$n = r(n) h'(r(n)). \tag{15}$$

On the other hand, according to (4), we have

$$H(s) = s u(s) - h(u(s)) = u(s) h'(u(s)) - h(u(s)),$$

since $s = h'(u(s))$. Hence,

$$B(n) = \max_{s>0}(n \ln s - H(s)) = \max_{s>0}[n \ln h'(u(s)) - u(s)h'(u(s)) + h(u(s))]$$

$$= \max_{u>0}[n \ln h'(u) - uh'(u) + h(u)].$$

The maximum condition is written in the form

$$n = u(n)h'(u(n)). \tag{16}$$

It is easy to see that equation (16) coincides with equation (15). Therefore one can write

$$B(n) = n \ln n - n \ln r(n) - n + h(r(n)). \tag{17}$$

Adding the functions $A(n)$ and $B(n)$, in accordance with (14) and (17), we obtain the identity (13).

Finally, for the series we have the estimate

$$\sum_{n=0}^{\infty} |f_n F_n| \leq \sqrt{2\pi}\, C_\varepsilon B \sum_{n=0}^{\infty} \sqrt{n}\, E_n (1+\varepsilon)^n (1-\delta)^n. \tag{18}$$

Since δ is a fixed number, while ε can be chosen small, the series (18) converges for any function $f(\zeta) \in Z_H$. At the same time we have obtained the boundedness of the functional (10) with respect to the norm $\|\cdot\|_p$ for $p > 1/\delta - 1$, which also means the boundedness of the functional (10) on the whole space Z_H.

By what has been proved, there exists a measure $\sigma(\zeta)$ such that

$$F_n = (-i)^n g_n n! = \int \zeta^n \, d\sigma(\zeta),$$

and moreover

$$\int |d\sigma(\zeta)| \exp\{H((1+1/p)|\zeta|)\} < \infty$$

for $p > 1/\delta - 1$. Consequently,

$$g_n = \int \frac{(i\zeta)^n}{n!} \, d\sigma(\zeta).$$

Multiplying by z^n and summing, we obtain on the left and on the right a convergent series, and finally

$$g(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \int \frac{(iz\zeta)^n}{n!}\, d\sigma(\zeta) = \int e^{iz\zeta}\, d\sigma(\zeta),$$

as was asserted.

In conclusion I express my deep gratitude to V. S. Vladimirov, A. F. Leont' ev, and A. Epshtein for useful discussions.

Joint Institute for Nuclear Research
Dubna, Moscow Region

Received
30 III 1970

REFERENCES

1. I. M. Gel′fand, G. E. Shilov, *Generalized Functions*, vol. 2, Moscow, 1958.
2. A. F. Leont′ev, UMN, 24, no. 2, 97 (1969).
3. M. A. Evgrafov, *Asymptotic Estimates and Entire Functions*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.