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Abstract

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V. P. MASLOV

PROPERTIES OF ABSOLUTE CONVERGENCE OF MULTIDIMENSIONAL FOURIER SERIES FROM THE POINT OF VIEW OF THE GEOMETRY OF WEAK DISCONTINUITIES OF THE EXPANDED FUNCTIONS

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It is obvious that the function $|x_1|^\alpha$ for $\alpha > 0$ expands into a uniformly absolutely convergent Fourier series in the n-dimensional cube $-\pi \le x_i \le \pi,\ i = 1, \dots, n$. The surface of weak discontinuity of such a function is the plane $x_1 = 0$. At the same time, it turns out that the Fourier series for the function $\big||x|^\alpha - 1\big|$, for $\alpha \ne 2k,\ \alpha < (n-1)/2$, weakly discontinuous on the sphere $|x| = 1$, does not converge uniformly absolutely. On the other hand, for this same function the series in the eigenfunctions of the first boundary-value problem for the Laplace operator in the spherical layer $1/2 \le |x| \le 2$ converges uniformly absolutely for $\alpha > 0$.

We shall pose the following question: what conditions must the surface of weak discontinuity of a function satisfy in order that the function expand into an absolutely convergent series with respect to at least one of the systems of eigenfunctions of an elliptic operator?* We shall first consider the double Fourier series, and then pass to the general case.

The trigonometric functions

$$\begin{matrix}\sin\\ \cos\end{matrix} nx_1 \cdot \begin{matrix}\sin\\ \cos\end{matrix} mx_2$$

serve as eigenfunctions of the Laplace–Beltrami operator on the torus. The corresponding eigenvalues are $n^2 + m^2$. Suppose that the function $\Phi(x)$ expands into an absolutely convergent series with respect to at least one of the systems of eigenfunctions of this operator. Then we shall say that the trigonometric Fourier series for $\Phi(x)$ converges quasi-absolutely. This agrees with the following other definition of quasi-absolute convergence and divergence.

Definition 1. We shall say that the Fourier series

$$\sum a_{nm} \frac{\sin}{\cos} nx_1 \cdot \frac{\sin}{\cos} mx_2$$

converges quasi-absolutely if the series

$$\sum_k \left| \sum_{n^2+m^2=k} a_{nm} \frac{\sin}{\cos} nx_1 \cdot \frac{\sin}{\cos} mx_2 \right|$$

converges, and that it diverges quasi-absolutely in a neighborhood of the point $x = x^0$ if the sequence of partial sums

$$\sum_{k=0}^{N} \left| \sum_{n^2+m^2=k} a_{nm} \frac{\sin}{\cos} nx_1 \cdot \frac{\sin}{\cos} mx_2 \right|$$

is unbounded in every neighborhood of the point $x = x^0$.

We shall now define a class of functions weakly discontinuous on a curve.

Definition 2. Let $M' \subset R^2$ be a one-dimensional, infinitely smooth submanifold, periodic (with period 2π), without boundary.** We shall

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* Systems of eigenfunctions differ by the choice of orthonormal bases in the eigensubspaces.

** Not necessarily closed and not necessarily connected, so that in the square $-\pi \leq x_1, x_2 \leq \pi$ it may be, for example, a closed curve or an interval.

say that the periodic function $\Phi(x)$ **belongs to the class** $\mathcal{L}^{\beta\alpha}(M^1)$ if: 1) it is sufficiently smooth in the domain $R^2 \setminus M^1$; 2) its restriction to M^1 is sufficiently smooth; 3) along the normal to M^1 it belongs to the Sobolev class $W_2^\alpha[R^1]$ $(\alpha > 1/2)$; 4) along the normal to M, at no point of M^1 does it belong to the Hölder class C^β $(\beta \geq \alpha - 1/2)$.

Denote by $S_1(M^1)$ the evolute (the geometric locus of centers of curvature) of the submanifold M^1, and by $S_k(M^1)$ the set of points of the evolute $S_1(M^1)$ corresponding to those points of M^1 at which $k-1$ derivatives of the curvature vanish. Thus, $S_{k+1}(M^1) \subset S_k(M^1)$.

Theorem 1. Let $\Phi(x) \in \mathcal{L}^{\beta\alpha}(M^1)$, with $\alpha > \dfrac{k}{k+1}$, $\beta \leq \dfrac{1}{2}\dfrac{k}{k+2}$, where $k > 0$ is some integer. Then in a neighborhood of the set $S_k(M^1)$ the Fourier series for $\Phi(x)$ diverges quasi-absolutely.

Corollary 1. The Fourier series for $\Phi(x) \in \mathcal{L}_{1/6}^{1/2+\delta}(M^1)$ can converge uniformly quasi-absolutely only when M^1 is an interval of a straight line.

Corollary 2. The Fourier series for $\Phi(x) \in \mathcal{L}^1_{1/2}(M^1)$, where M^1 is an analytic submanifold, can converge uniformly quasi-absolutely only when no connected component of M^1 lies on a circle.

Remark. It is known (1) that if the derivative $\Phi(x)$, $x \in R^2$, is piecewise continuous, then the Fourier series for $\Phi(x)$ converges uniformly absolutely. In this case $\Phi(x) \in \mathcal{L}^{3/2-\delta}_{1+\varepsilon}(M^1)$, $\delta > 0$, $\varepsilon > 0$.

Example. Let $\Phi(x) = \varphi(x)(f(x))^\alpha$, where $\varphi(x)$ is a finite function in the square $-\pi \le x_1,\ x_2 \le \pi$, and $f(x) \in C^\infty$. Extending the set $\{(x : f(x) = 0) \cap \operatorname{supp} \varphi(x)\}$ periodically, we obtain M^1. If

$$\frac{1}{2}\frac{k-1}{k+2} < \alpha < \frac{1}{2}\frac{k}{k+2},$$

then the Fourier series for $\Phi(x)$ in a neighborhood of $S_k(M^1)$ diverges uniformly quasi-absolutely.

We shall not formulate here the results for the three-dimensional case. Let us only note that for $n = 3$, for functions having a weak discontinuity on a surface (or on a curve), the role of the set S_1 will be played by the envelope surface of the two-parameter family of normals drawn to the surface (or curve) of discontinuity, and the role of the set S_2 by special points of the envelope surface.

For the general case we shall formulate here only some sufficient conditions for absolute divergence.

Sufficient conditions for partial absolute convergence will be given in another paper.

Consider on a manifold M^n (not necessarily compact, in particular on R^n) a positive, self-adjoint pseudodifferential elliptic operator $\hat{L}$ of order m with symbol* $L(p, x)$, where x, p is the space of all cotangent vectors to M^n. Let $\{\psi_k(x)\}$ be some system of eigenfunctions of the operator $\hat{L}$ (the spectrum of $\hat{L}$ is assumed discrete).

Definition 3. The series $\sum_k a_k \psi_k(x)$ is said to **diverge absolutely in a neighborhood of the point** $x = \tilde{x}$ if the sequence of partial sums

$$\sum_{k=1}^{N} |a_k|\,|\psi_k(x)|$$

is unbounded in every neighborhood of the point $x = \tilde{x}$.

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* Here and below the principal part of the full symbol is meant, i.e. the homogeneous function of p of order m. As usual, smoothness and boundedness of $L(p, x)$ with respect to the variables $x \in M^n$ are assumed.

Definition 4. Let $M^{n-1} \subset M^n$ be a smooth submanifold of M^n. We shall call a function $\Phi(x)$ **weakly discontinuous of order** a **on** M^{n-1} if: 1) $\Phi(x)$ does not belong to C^a at any point $x \in M^{n-1}$; 2) $\Phi(x)$ is sufficiently smooth in the domain $M^n \setminus M^{n-1}$; 3) the restriction of $\Phi(x)$ to M^{n-1} is sufficiently smooth; 4) $\Phi(x)$ has compact support.

Consider on M^n the family of trajectories $x(t)$ defined as the solution of the Hamiltonian system

$$\dot{x} = \frac{\partial}{\partial p} L(p, x), \qquad \dot{p} = -\frac{\partial}{\partial x} L(p, x), \tag{1}$$

satisfying the initial conditions

$$x|_{t=0} = x^0 \in M^{n-1}, \qquad p|_{t=0} = p^0, \tag{2}$$

where p^0 is orthogonal to every tangent vector to M^{n-1} and is such that

$$L(p^0, x^0) = 1. \tag{3}$$

The solution of problem (1)–(3) exists globally.

Theorem 2. *The series obtained in the expansion of a function $\Phi(x)$, weakly discontinuous of order $1/6$ on M^{n-1}, in the eigenfunctions of the operator $\hat{L}$, diverges absolutely in a neighborhood of the focal points of the family of trajectories $x(t)$ of problem (1)–(3).*

We outline the proof of Theorem 1. Consider the equation

$$i\,\partial u/\partial t = \sqrt[m]{\hat{L}}\,u. \tag{4}$$

As is known ([2]), $\sqrt[m]{\hat{L}}$ is a pseudodifferential operator with symbol $\sqrt[m]{L(p,x)}$. Therefore (4) is a pseudohyperbolic equation on M^n. Let us note that the results ([3,4]), formulated for Euclidean space, automatically carry over to pseudohyperbolic equations on manifolds. We shall apply them in our case.

Set for (4) the Cauchy problem:

$$u|_{t=0} = \Phi(x), \qquad x \in M^n.$$

Now suppose that the Fourier series for $\Phi(x)$ with respect to some system of eigenfunctions $\{\psi_k\}$ of the operator $\hat{L}$ converges absolutely, i.e.

$$\Phi(x) = \sum a_k \psi_k(x),$$

and, moreover,

$$\sum |a_k|\,|\psi_k(x)| < \text{const}$$

in a neighborhood of the focal points. Hence

$$|u(x,t)| = \left|\sum a_k \exp\left(i \sqrt[m]{\lambda_k}\, t\right) \psi_k(x)\right| \le \sum |a_k|\,|\psi_k(x)| < \text{const}, \qquad (5)$$

where λ_k are the eigenvalues of the operator $\hat{L}$. But the left-hand side of inequality (5) tends to ∞ in a neighborhood of the focal points of the solution $x(t)$ of the system

$$\dot{x} = \frac{\partial}{\partial p} \sqrt[m]{L(p,x)}, \qquad \dot{p} = -\frac{\partial}{\partial x} \sqrt[m]{L(p,x)},$$

satisfying the same conditions (2)–(3), by virtue of ([3]). Since the focal points of this problem coincide with the focal points of the solution $x(t)$ of problem (1)–(3), we have arrived at a contradiction, as was required to prove.

Now consider functions $\Phi(x)$ of class $^1/_4$ on M^n with weak discontinuities. Let $T_{2n}(M^n)$ be the cotangent (phase) space x, p over M^n, and let $K^n \to T_{2n}(M^n)$ be the natural regular mapping of $M^{n-1} \times R \approx K^n$ into the manifold $T_{2n}(M^n)$ onto the tube formed by the trajectories $x(t), p(t)$, the solutions of problem (1)–(3).

Analogously to Theorem 2, one proves

Theorem 3. The series in eigenfunctions of the operator $\hat{L}$ of a function $\Phi(x)$, weakly discontinuous on M^{n-1}, of order $^1/_4$, converges absolutely in a neighborhood of the special points of the image $S^{n-1} \subset M^n$ of the cycle of singularities of the projection
$\pi : K^n \to M^n$.

Moscow Institute
of Electronic Machine Building

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Note: Figure translations are in progress. See original paper for figures.