HEYMAN' S REGULARITY THEOREM FOR THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Abstract

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MATHEMATICS

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HEYMAN' S REGULARITY THEOREM FOR THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Let S be the class of functions $f(z) = z + c_2 z^2 + \cdots$, regular and univalent in the unit disk, and let Σ be the class of functions $F(z) = z + a_0 + a_1 z^{-1} + \cdots$, meromorphic and univalent in the domain $|z| > 1$. Denote the expansions:

$$\ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} 2\gamma_k z^k, \qquad f(z) \in S; \tag{1}$$

$$\ln \frac{z-t}{F(z)-F(t)} = \sum_{k=1}^{\infty} A_k(t) z^{-k}, \qquad F(z) \in \Sigma; \tag{2}$$

$$(1-z)^{-h} = \sum_{k=0}^{\infty} d_k(h) z^k, \qquad h > 0. \tag{3}$$

In addition, by the symbol $\{\psi(z)\}_n$ we shall denote the coefficient of z^n in the expansion of the function $\psi(z)$ about $z = 0$, and by $M(r, f)$ the maximum of $|f(z)|$ on the circle $|z| = r,\ 0 < r < 1$.

It is known ([1]) that for every function $f(z) \in S$ there exists

$$\lim_{r\to 1} (1-r)^2 M(r, f) = \alpha, \quad 0 \le \alpha \le 1,$$

and, for $\alpha > 0$, a radius of greatest growth of the function, i.e. such a $\varphi_0 \in [0, 2\pi]$ that

$$\lim_{r\to 1} (1-r)^2 |f(re^{i\varphi_0})| = \alpha, \qquad 0 < \alpha \le 1. \tag{4}$$

Using (4) and the "area inequality" ([2]):

$$\sum_{n=1}^{\infty} n|A_n(z)|^2 \le \ln \frac{1}{1-r^2}, \qquad |z| = \frac{1}{r} > 1,$$

I. E. Bazilevich (3) proved for $f(z) \in S$ the inequality

$$\sum_{k=1}^{\infty} k \left| \gamma_k - \frac{1}{k} e^{ik\varphi_0} \right|^2 \le \frac{1}{2} \ln \frac{1}{\alpha}. \tag{5}$$

It is proved below that conditions (4) and (5) are already sufficient for the regular growth of the Taylor coefficients even of non-univalent functions, i.e. the following holds.

Theorem. For a function $f(z) = z + c_2 z^2 + \cdots$ satisfying conditions (4) and (5), for any $\lambda > 1/4$ the asymptotic equality

$$\frac{\{(f(z)/z)^{\lambda}\}_n}{d_n(2\lambda)} \sim \alpha^{\lambda} \exp\{i[\lambda \arg f(re^{i\varphi_0}) - (n+\lambda)\varphi_0]\} \qquad (n \to \infty), \tag{6}$$

holds, where r is connected with n by the relation

$$1 - r = \theta/n, \qquad m < \theta < M \tag{7}$$

(m and M are positive constants).

The proof of the theorem is based on two lemmas.

Lemma 1. If the coefficients of the power series $\sum_{k=1}^{\infty} A_k z^k$ satisfy–

satisfy the conditions:

$$\text{a)} \quad \sum_{k=1}^{\infty} k|A_k|^2 < \infty;$$

$$\text{b)} \quad \operatorname{Re} \sum_{k=1}^{n} A_k = O(1) \quad (n \to \infty) \tag{8}$$

and if the power series is formed

$$\sum_{k=0}^{\infty} D_k z^k = \exp\left(\sum_{k=1}^{\infty} A_k z^k \right), \tag{9}$$

then for every $h > 1/2$ the following asymptotic equality holds:

$$\sum_{k=0}^{n} D_k - \frac{1}{d_n(h)} \sum_{k=0}^{n} d_{n-k}(h)\overline{D}_k = o(1) \quad (n \to \infty). \tag{10}$$

Denoting by $S_n^{(h)}$ the numerator of the Cesàro mean in (10), i.e.,

$$S_n^{(h)} = \sum_{k=0}^{n} d_{n-k}(h) D_k = \left\{ \frac{1}{(1-z)^h} \exp\left(\sum_{k=1}^{\infty} A_k z^k \right) \right\}_n \quad (n = 0, 1, \dots), \tag{11}$$

we write for the Cesàro means the obvious identity

$$\frac{S_n^{(h)}}{d_n(h)} - \frac{S_n^{(h+1)}}{d_n(h+1)} = \sum_{k=0}^{n-1} \left[\frac{d_k(h)}{d_n(h)} - \frac{d_k(h+1)}{d_n(h+1)} \right] D_{n-k} = \sum_{k=1}^{n} d_{n-k}(h) \cdot k D_k/(n+h) \times$$

$$\times d_n(h) = \sum_{k=1}^{n} d_{n-k}(h) \sum_{\nu=1}^{k} D_{k-\nu} \nu A_\nu \big/ (n+h) d_n(h) = \sum_{k=1}^{n} S_{n-k}^{(h)} k A_k/(n+h) d_n(h). \tag{12}$$

Cauchy' s inequality, applied to the right-hand side of (12), gives the estimate

$$\left| \frac{S_n^{(h)}}{d_n(h)} - \frac{S_n^{(h+1)}}{d_n(h+1)} \right| \le \frac{1}{(n+h) d_n(h)} \left(\sum_{k=0}^{n-1} |S_k^{(h)}|^2 \sum_{k=1}^{n} k^2 |A_k|^2 \right)^{1/2}. \tag{13}$$

In order to estimate $\sum_{k=0}^{n-1} |S_k^{(h)}|^2$, we note that, by the conditions of the lemma, the relation

$$\sum_{k=1}^{n} k \left| \frac{h}{k} + A_k \right|^2 = h^2 \sum_{k=1}^{n} \frac{1}{k} + 2h \operatorname{Re} \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} k |A_k|^2 \le h^2 \sum_{k=1}^{n} \frac{1}{k} + O(1)$$

holds, and, consequently, as $n \to \infty$,

$$\delta_n(h) = \max_{1 \le \nu \le n} \left\{ \frac{1}{h^2} \sum_{k=1}^{\nu} k \left| \frac{h}{k} + A_k \right|^2 - \sum_{k=1}^{\nu} \frac{1}{k} \right\} = O(1). \tag{14}$$

But earlier, for the coefficients of a compound function of exponential form, inequality (4) was established:

$$\sum_{k=0}^{n-1} \left| \left\{ \exp\left(\sum_{k=1}^{\infty} \left(\frac{h}{k} + A_k \right) z^k \right) \right\}_k \right|^2 \leq \exp(h\delta_{n-1}(h)) \sum_{k=0}^{n-1} d_k^2(h),$$

using which, and taking into account (11) and (14), we shall have

$$\sum_{k=0}^{n-1} |S_k^{(h)}|^2 = O(1) \sum_{k=0}^{n-1} d_k^2(h) \quad (n \to \infty), \quad h > 0. \tag{15}$$

Simple calculations give, for the sum of squares of binomial coefficients when $h > 1/2$, the estimate

$$\sum_{k=0}^{n-1} d_k^2(h) = O(1) n^{2h-1} \quad (n \to \infty),$$

which, together with (15), (13), and condition (8a), leads to the result

$$S_n^{(h)}/d_n(h) - S_n^{(h+1)}/d_n(h+1) = o(1) \quad (n \to \infty), \qquad h > \frac{1}{2}. \tag{16}$$

In particular, for $h = 1$, it follows from (16) that

$$\frac{1}{n} \sum_{k=1}^{n} kD_k = o(1) \quad (n \to \infty). \tag{17}$$

Now consider the difference

$$\sum_{k=0}^{n} D_k - \frac{S_n^{(h)}}{d_n(h)} = \frac{1}{d_n(h)} \sum_{k=1}^{n} \frac{1}{k} \left[d_n(h) - d_{n-k}(h) \right] kD_k. \tag{18}$$

Since for $h \geq 2$ the sequence of numbers

$$\frac{1}{k} \left[d_n(h) - d_{n-k}(h) \right]$$

$(k = 1, 2, \ldots, n)$ is nondecreasing, by Abel' s inequality, from (18) and (17) we obtain

$$\sum_{k=0}^{n} D_k - \frac{S_n^{(h)}}{d_n(h)} = o(1) \quad (n \to \infty), \qquad h \geq 2. \tag{19}$$

Equalities (16) and (19), considered together, prove the lemma completely.

Lemma 2. Under the conditions of Lemma 1, the following asymptotic equalities hold for $h \geq \frac{1}{2}$:

$$\frac{1}{d_n(h)}\sum_{k=0}^{n} d_{n-k}(h) D_k \sim \sum_{k=0}^{\infty} D_k r^k \sim \exp\left(\sum_{k=1}^{n} A_k\right) \quad (n \to \infty), \tag{20}$$

where r is connected with n by formula (7).

Taking into account Lemma 1 and condition (8b), it suffices to prove the relations

$$\sum_{k=0}^{n} D_k \sim \sum_{k=0}^{\infty} D_k r^k \sim \exp\left(\sum_{k=1}^{n} A_k\right) \quad (n \to \infty). \tag{21}$$

We first derive the second part of (21). To this end consider the difference

$$\sum_{k=1}^{n} A_k - \sum_{k=1}^{\infty} A_k r^k = \sum_{k=1}^{n} A_k (1 - r^k) - \sum_{k=n+1}^{\infty} A_k r^k.$$

Applying Cauchy' s inequality, we have

$$\left|\sum_{k=1}^{n} A_k - \sum_{k=1}^{\infty} A_k r^k\right| \leq \left[\sum_{k=1}^{n} k|A_k|^2 (1 - r^k) \sum_{k=1}^{n} \frac{1}{k}(1 - r^k)\right]^{1/2} +$$

$$+\frac{1}{n}\left[\sum_{k=n+1}^{\infty} k|A_k|^2 \cdot \sum_{k=n+1}^{\infty} k r^{2k}\right]^{1/2} \leq \left[\sum_{k=1}^{n} k|A_k|^2 (1 - r^k)\right]^{1/2} [n(1-r)]^{1/2} +$$

$$+\left[\sum_{k=n+1}^{\infty} k|A_k|^2\right]^{1/2} \frac{1}{n(1-r)},$$

whence, taking (7) and (8a) into account, we conclude that

$$\sum_{k=1}^{n} A_k - \sum_{k=1}^{\infty} A_k r^k = o(1) \quad (n \to \infty), \tag{22}$$

which, after exponentiation, gives the second part of (21).

Next, consider the other difference for $0 < r < 1$:

$$\sum_{k=0}^{n} D_k - \sum_{k=0}^{\infty} D_k r^k = \sum_{k=0}^{n} D_k (1 - r^k) - \sum_{k=n+1}^{\infty} D_k r^k. \tag{23}$$

The first sum on the right-hand side of (23) is estimated with the aid of Abel' s inequality, namely

$$\left|\sum_{k=0}^{n} D_k (1 - r^k)\right| = \left|\sum_{k=1}^{n} k D_k \frac{1}{k}(1 - r^k)\right| \leq (1 - r) \max_{1 \leq \nu \leq n} \left|\sum_{k=1}^{\nu} k D_k\right|. \tag{24}$$

From (24), (7), and (17) we obtain

$$\sum_{k=0}^{n} D_k (1 - r^k) = o(1) \qquad (n \to \infty). \tag{25}$$

The second sum can be estimated, also using Abel' s transformation:

$$\left|\sum_{k=n+1}^{N} D_k r^k\right| \leq \left(\sum_{k=n+1}^{N} \frac{1}{k} r^k + \frac{2n+1}{n+1} r^{n+1}\right) \max_{n+1 \leq k \leq N} \frac{1}{k} \left|\sum_{\nu=n+1}^{k} \nu D_\nu\right|. \tag{26}$$

If r is chosen according to (7), then the first factor in (26) is uniformly bounded with respect to n and N, while the second factor, as $n \to \infty$, tends to zero uniformly with respect to N by (17), and, consequently,

$$\sum_{k=n+1}^{\infty} D_k r^k = o(1),$$

which, together with (25) and (86), gives the first part of (21). The lemma is proved.

Proof of the theorem. Without loss of generality one may assume that $\varphi_0 = 0$, since otherwise, instead of $f(z)$, it suffices to consider $e^{-i\varphi_0} f(e^{i\varphi_0} z)$.

Starting from the identity

$$\left(\frac{f(z)}{z}\right)^\lambda = \frac{1}{(1-z)^{2\lambda}} \left[\frac{f(z)}{z}(1-z)^2\right]^\lambda \tag{27}$$

and denoting

$$\ln \left[\frac{f(z)}{z}(1-z)^2\right]^\lambda = \sum_{k=1}^{\infty} A_k z^k, \qquad A_k = 2\lambda \left(\gamma_k - \frac{1}{k}\right),$$

$$\left[\frac{f(z)}{z}(1-z)^2\right]^\lambda = \exp\left(\sum_{k=1}^{\infty} A_k z^k\right) = \sum_{k=0}^{\infty} D_k z^k, \tag{28}$$

we write the equality of the Taylor coefficients for the functions in (27):

$$\left\{ \left(\frac{f(z)}{z} \right)^{\lambda} \right\}_n = \sum_{k=0}^{n} d_{n-k}(h) D_k \qquad (n = 0, 1, ...), \qquad h = 2\lambda. \tag{29}$$

It is not difficult to verify that the coefficients of the power series $\sum_{k=1}^{\infty} A_k z^k$ satisfy conditions (8). Therefore, for $h > 1/2$ and for r chosen according to (7), by Lemma 2 we shall have:

$$\frac{1}{d_n(h)} \sum_{k=0}^{n} d_{n-k}(h) D_k \sim \sum_{k=0}^{\infty} D_k r^k = \left[|f(r)| \frac{(1-r)^2}{r} \right]^{\lambda} \exp(i\lambda \arg f(r)),$$

which, together with (29) and (4), proves the theorem.

From the theorem just obtained there follows directly Hayman' s regularity theorem [1] for the coefficients of univalent functions in the case $\alpha > 0$ (proved by him for p-valent functions). If $\alpha = 0$, then the regularity theorem is proved more elementarily.

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Note: Figure translations are in progress. See original paper for figures.