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## Abstract

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*MATHEMATICS*

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# ON ONE VARIANT OF AVERAGING IN INTEGRO-DIFFERENTIAL EQUATIONS

*(Presented by Academician A. A. Dorodnitsyn on 17 XI 1969)*

**1.** Consider the system of nonlinear integro-differential equations

$$\dot{x} = \varepsilon X\left(t, x, \int_0^t \varphi(t, s, x(s))\, ds\right). \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, $x = (x_1, x_2, \dots, x_n)$.

In papers ($^1$,$^2$ ), for equations of the form (1) the following averaging procedure was proposed. Let

$$\psi(t, x) = \int_0^\infty \varphi(t, s, x)\, ds, \tag{2}$$

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x))\, dt. \tag{3}$$

Then to the system (1) there is put in correspondence an averaged system of the form

$$\dot{\xi} = \varepsilon X_0(\xi). \tag{4}$$

In the present note theorems are proved on the closeness of the solutions of systems (1) and (4), both on a finite and on an infinite interval.

**Theorem 1.** Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$ be defined and continuous in the domain $Q\{t \geq 0,\ s \geq 0,\ x \in D,\ y \in E_n\}$, and suppose that in this domain the following conditions are fulfilled:

1)
$$|X(t,x',y') - X(t,x'',y'')| \leq \lambda\{|x'-x''| + |y'+y''|\},$$

$$|\varphi(t,s,x') - \varphi(t,s,x'')| \leq \mu(t,s)|x'-x''|,$$

$$\int_0^t d\tau \int_0^\tau \mu(\tau,s)\,ds \leq At^\alpha, \qquad A>0, \quad 0 \leq \alpha < 1, \quad \lambda = \text{const}.$$

2) The function $\psi(t,x)$, defined by equality (2), satisfies the Lipschitz condition

$$|\psi(t,x') - \psi(t,x'')| \leq \nu|x'-x''|, \qquad \nu = \text{const}.$$

3) At each point $x$ of the domain $D$ the limit (3) exists, and the function $X_0(x)$ is bounded ($|X_0| \leq M$) and satisfies the Lipschitz condition.

4) The solution $\xi = \xi(t)$, $\xi(0) = x(0) \in D$, of the averaged system is defined for all $t \geq 0$ and lies in the domain $D$ together with some $\rho$-neighborhood.

5) Along the trajectory $\xi(t)$,

$$\int_0^t d\tau \left| \int_\tau^\infty \varphi(\tau,s,\xi(\tau))\,ds \right| \leq Bt^\beta, \qquad B>0, \quad 0 \leq \beta < 1.$$

Then for any $\eta > 0$ and $L > 0$ one can specify an $\varepsilon_0$ such that, for $\varepsilon < \varepsilon_0$, on the interval $0 \leqslant t \leqslant L\varepsilon^{-1}$ the inequality

$$|x(t) - \xi(t)| < \eta$$

will hold.

**Proof.** As in (3), it is shown that for any $a > 0$ one can specify an $\bar{\varepsilon}$ such that, for $\varepsilon < \bar{\varepsilon}$, on the interval $0 \leqslant t \leqslant L\varepsilon^{-1}$ the inequality

$$\varepsilon \left| \int_0^t \left[ X\left(\tau, \xi(\tau), \int_0^\infty \varphi(\tau,s,\xi(\tau))\,ds\right) - X_0(\xi(\tau)) \right] d\tau \right| < a$$

will hold. Hence,

$$|x-\xi| \leqslant a + \varepsilon\lambda \int_0^t |x(\tau) - \xi(\tau)|\,d\tau + \varepsilon\lambda \int_0^t d\tau \int_0^\tau \mu(\tau,s)|x(s) - \xi(s)|\,ds$$
$$+ \varepsilon\lambda \int_0^t d\tau \left| \int_\tau^\infty \varphi(\tau,s,\xi(\tau))\,ds \right| + \varepsilon\lambda \int_0^t d\tau \int_0^\tau \mu(\tau,s)|\xi(s) - \xi(\tau)|\,ds.$$

From this, taking into account the conditions of the theorem, we find on the interval $0 \leqslant t \leqslant L\varepsilon^{-1}$:

$$|x-\xi| \leqslant \left(a + \lambda MAL^{1+\alpha}\varepsilon^{1-\alpha} + \lambda BL^{1+\beta}\varepsilon^{1-\beta}\right) e^{\lambda L + AL\alpha\varepsilon^{1-\alpha}}.$$

The assertion of the theorem follows from the last inequality.

**Remark 1.** Consider the system

$$\dot{x} = \varepsilon X(t,x) + \varepsilon \int_0^t Y(t,s,x(s))\, ds. \tag{*}$$

If the functions $X$ and $Y$ are bounded, then the solution $x(t)$ of system $(*)$ will change substantially already on an interval of order $\varepsilon^{-1/2}$. Therefore, in particular, for systems of the form $(*)$ one can formulate an averaging theorem analogous to Theorem 1, establishing the closeness of the solutions of the original and averaged systems on the interval $0 \leqslant t \leqslant L\varepsilon^{-1/2}$. In this case the parameters $\alpha$ and $\beta$ will vary within the limits $0 \leqslant \alpha < 2$, $0 \leqslant \beta < 2$.

**Theorem 2.** Replace condition (3) of Theorem 1 by the following:

a) at every point $x$ of the domain $D$, uniformly with respect to $t$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} X(t,x,\psi(t,x))\, dt = X_0(x), \tag{5}$$

and the function $X_0(x)$ is bounded and satisfies the Lipschitz condition;

b) the solution $\xi = \xi(\tau)$, $\tau = \varepsilon t$, of the averaged system

$$\dot{\xi} = \varepsilon X_0(\xi), \qquad \xi(0) = x(0)$$

is uniformly* asymptotically stable;

c) equation (1) has no singular points.

Then for any $0 < \eta < \rho$ one can specify an $\varepsilon_0$ such that, for $\varepsilon < \varepsilon_0$, for all $t > 0$ the inequality

$$|x(t) - \xi(t)| < \eta$$

will hold.

**Proof.** The proof is carried out by the methods set forth in (4, 6).

* For the notion of uniform asymptotic stability, see, for example, (5).

**Remark 2.** Let us note that in many cases the requirement of uniform asymptotic stability appearing in Theorem 2 can be weakened.

**2.** Consider a system of a more general form

$$\dot{x} = \varepsilon X\left(t, x, \dot{x}, y, \dot{y}, \int_0^t \varphi(t,s,x(s),\dot{x}(s),y(s),\dot{y}(s))\, ds\right),$$

$$\dot{y} = Y_0(t,x,y) + \varepsilon Y_1\left(t, x, \dot{x}, y, \dot{y}, \int_0^t \psi(t,s,x(s),\dot{x}(s),y(s),\dot{y}(s))\, ds\right). \tag{6}$$

If the general solution of system (6) for $\varepsilon = 0$ is known, then, as was shown in (4), this system can be reduced to standard form. After this, the averaging procedure under consideration can be applied to the resulting system.

**3.** Let a system of nonlinear integral equations of Volterra type be given:

$$u(t) = \varepsilon \int_0^t \Phi(t, s, u(s))\, ds. \tag{7}$$

Differentiating (7), we find

$$\dot{u} = \varepsilon \Phi(t, t, u) + \varepsilon \int_0^t \frac{\partial \Phi(t, s, u(s))}{\partial t}\, ds. \tag{8}$$

System (8) is integro-differential, and averaging Theorems 1 and 2 are applicable to it.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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