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A THEOREM ON EXTRACTING THE BASIC SYSTEM FROM THE VECTOR SYSTEM BY ONE PEAK OF MULTIPLICITY n

(GENERAL CASE, FEDOROV GROUP $P1$)

Let an acentric basic system (b.s.) of N points (Fig. 3) contain n pairs of points for which the condition

$$\mathbf{r}_{12} = \mathbf{r}_{34} = \dots = \mathbf{r}_{2n-1,2n}, \tag{1}$$

is satisfied, but among which there are no three or more pairs satisfying the condition

$$\begin{aligned} \mathbf{r}_{13} = \mathbf{r}_{24} = \dots = \mathbf{r}_{2p-2,2p} = \dots = \mathbf{r}_{14} = \dots = \mathbf{r}_{2m-3,2m} = \\ = \mathbf{r}_{15} = \dots = \mathbf{r}_{2n-q,\,2n}, \end{aligned} \tag{2}$$

where $q = 2, 3, \dots, 2n-1$.

In other words, the indicated n pairs of points of the b.s. (see condition (1)) must be separated from one another by unequal distances. Then in the corresponding

Fig. 1. V.s. corresponding to the b.s. of Fig. 3. (The basic system consists of $N = 9$ points, for which conditions (1) and (2) are fulfilled. The $2n$ points are combined into $n = 4$ pairs.) Lines parallel to the line passing through the origin are highlighted. Centrosymmetrically related centers are highlighted in the same way. Double vectors {II—VII and (−II) ÷ (−VII)} and vector I–quadruple–are drawn.

vector system (v.s.) two peaks of multiplicity n arise at the ends of the line passing through the origin, and these peaks are related by the inversion point at the ori-

initially. The remaining peaks of the v.s. must, by virtue of condition (2), be no more than double*.

For such a v.s. one can formulate the following theorem:

*In order to extract the o.s. from the vector set by one peak of multiplicity n, it is necessary and sufficient that n be equal to three***.* That is, it can be shown that, in order to extract from such a v.s. the entire o.s., it is enough, in the corresponding algorithm, to use only the vectors between three arbitrary pairs out of the n pairs of the o.s.; and this number of vectors also proves sufficient for extracting the o.s. In other words, the excess of the peak multiplicity over three may be disregarded.

Fig. 2. $2n$ copies of the o.s. (when the v.s. is shifted by vector I); half-length segments and the center of symmetry midway along vector I are shown

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The proof is simplest by contradiction. Suppose that in the v.s. there have been selected $n(n-1)+1$ segments (according to ([2,3])), combined into $(n-1)n/2$ parallelograms. As the initial object at the first stage of the extraction algorithm, choose a peak of multiplicity n at the end of the segment passing through the origin (I in Fig. 1). We obtain $2n$ copies of the o.s.: n direct and n inverted (Fig. 2). Among the points of these $2n$ copies there are also $n(n-1)+1$ segments, but now of half length ($= r_{12} = ... = r_{2n-1,2n}$); the segments are connected in pairs by a center of symmetry in the middle of shift vector I (Fig. 2). The indicated half-length segments are combined into $n(n-1)$ parallelograms with sides $\neq$ vector I and vectors II-VII***. Let $n = 2$; then at the second stage any of the vectors II-VII may be taken as vector II (but, according to ([4]), only one), and in accordance with ([4]), by shifting all points of Fig. 3—the image of vector I—by vector II, we always isolate two copies of the o.s.—direct and inverted (Fig. 3a; the center of symmetry is located at the center of the parallelogram constructed on vectors I and II ([4]); Fig. 3b–I and IV, Fig. 3c–I and V). It is impossible to get rid "automatically" of the extra copy within $P1$ for $n = 2$; for this it is necessary to carry out one more stage of extracting copies of the o.s. among the points of Fig. 2 by any of the vectors (IV, V, –III, –IV in Fig. 3a) from the origin to the segment remaining when the copies of Fig. 2 are shifted by vector II ([1]). However, such a segment arises only for $n = 3$, by virtue of conditions (5) and (5′) according to ([1]), but is absent for $n = 2$. Thus, the peak multiplicity must be at least equal to three.

* For the time being we do not consider the case where the remaining $N - 2n$ points of the o.s., among themselves and with the $2n$ points combined into pairs, can generate peaks of multiplicities n_1, n_2, etc.

** An algorithm for extracting the o.s. from the v.s. by a peak of multiplicity three was proposed in ([1]).

*** Any of the vectors satisfying condition (2).

We shall show that a peak multiplicity $n = 3$ is sufficient for isolating the entire b.s.

Let the peak multiplicity be $n \geq 4$; then, after shifting by vectors I and II, among the points of two copies of the b.s. from $n(n-1)+1$ half-lines there remain $2(n-1)$ half-lines*, which are connected in pairs by the point of inversion at the center of the parallelogram (Fig. 3a). By ([1], [5]) these $(n-1)$ half-lines will be independent. Excluding from consideration the doubled half-lines (they have already been used earlier), at the third stage we fix, as the shift vector, the vector (from the origin) to any of the remaining $(n-1)-1=(n-2)$ half-lines (Fig. 3b, c). A simple combination of the results of the second and third stages leaves **only one b.s.** ([1]). Since the $(n-2)$ half-lines are independent, the final result will not be determined by which of the half-lines is connected at the third stage. Thus, to isolate the b.s. from the v.s., it is sufficient, out of $\{n(n-1)+1\}$ half-lines, to use only three independent ones (and without the one passing through the origin—two), arising for a peak of multiplicity three, and thereby the theorem is proved.

The theorem proved makes it possible to formulate three consequences that are important in practice:

1) In a v.s., a peak of any multiplicity may always be regarded as threefold**.
2) To isolate the b.s. from a vector set, when it contains only one peak of multiplicity $n \geq 2$**, it is necessary and sufficient to carry out only three stages of isolation in all.
3) Having isolated in the v.s. $\{(n-1)n+1\}$ half-lines ([2]) and found among them $(n-1)$ independent ones, in the isolation algorithm the first should be taken to be the one passing through the origin, the second may be chosen arbitrarily, and the third (connected with the first two by relations (5) or (5′) from ([1])) among the half-lines on two copies of the b.s. that were isolated in the successive searches for the images of vectors I and II.

Fig. 3. Two copies of the b.s., obtained from points already isolated by vector I: **a**—after shifting by vector II; **b**—after shifting by vector V (from the origin to the half-line remaining after the shift by vector II); **c**—after shifting by vector IV (from the origin to a half-line also fixed during the shift by vector II). The points of the b.s. remaining when the results of the second and third stages are combined are marked by circles.

Figs. 1-3 illustrate the theorem for a b.s. $N=9$, $n=4$.

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* Strictly speaking, $2n$ half-lines should remain, corresponding to the number of pairs of points, but four half-lines merge pairwise on the sides of the parallelogram constructed on the shift vectors I and II.

** When conditions (1) and (2) are fulfilled.

Note: Figure translations are in progress. See original paper for figures.