

A NONLINEAR ADDITION TO THE LAGRANGIAN DENSITY OF THE GRAVITATIONAL FIELD AND COSMOLOGICAL SOLUTIONS WITHOUT A SINGULARITY

MATHEMATICAL PHYSICS

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.32133>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 531.5

MATHEMATICAL PHYSICS

V. Ts. GUROVICH

A NONLINEAR ADDITION TO THE LAGRANGIAN DENSITY OF THE GRAVITATIONAL FIELD AND COSMOLOGICAL SOLUTIONS WITHOUT A SINGULARITY

(Presented by Academician Ya. B. Zel'dovich, May 19, 1970)

The presence of singularities in solutions of Einstein's equations has recently stimulated the study of equations of higher order. The reason for this was the work of A. D. Sakharov ⁽¹⁾, in which arguments were given in favor of a nonlinear, in the four-curvature, addition to the Lagrangian density of the gravitational field.

Such a Lagrangian density corresponds to fourth-order equations for the components of the metric tensor, as was pointed out in ⁽²⁾. Such equations could contain cosmological solutions with a continuous transition from contraction to expansion. This possibility was noted in ⁽³⁾ and studied in detail in ⁽⁴⁾ for the Lagrangian density $L = R + aR^2$ within the framework of a homogeneous and isotropic model of the universe with flat accompanying space and the equation of state of matter $p = \varepsilon/3$. The gravitational equations obtained with the aid of such an L contain solutions with a regular transition from contraction to expansion at a finite density of matter (at the instant $t = 0$); however, they lead to a divergence of the scalar curvature R as $t \rightarrow +\infty$, or $t \rightarrow -\infty$.

To clarify the cause of the divergence of R , in work ⁽⁴⁾, for a cosmological model analogous to that considered in ⁽³⁾, a Lagrangian density of the form $L = R + aR^n$ was investigated. For the power $n = 4/3$, an exact solution was found that is regular for all values of t . Under the corresponding "initial" conditions, the solutions found as $t \rightarrow \infty$ tended asymptotically to the Friedmann solution for the corresponding model of the universe and $p = \varepsilon/3$. In the present work, on the basis of the equations obtained in ⁽⁴⁾, exact cosmological solutions are constructed for dustlike matter, corresponding to the three Friedmann models of the universe. Remaining regular at the instant of maximum contraction of matter (at $t = 0$), these solutions pass into the corresponding Friedmann solutions as $|t| \rightarrow \infty$, in contrast to the solution obtained in work ⁽⁴⁾, where the asymptotic approach to the Friedmann solution occurs as $t \rightarrow +\infty$, or $t \rightarrow -\infty$.

We consider an action for the gravitational field of the form

$$S = A \int [R + l^{-2} f(l^2 R)] \sqrt{-g} d\Omega. \quad (1)$$

Variation of (1), together with the action for matter, as shown in (4), leads to the gravitational equations

$$R_i^k - \frac{1}{2} \delta_i^k R + l^{-2} \left[\frac{\partial f}{\partial R} R_i^k - \frac{1}{2} \delta_i^k f + (\delta_i^k g^{lm} - \delta_i^l g^{km}) \left(\frac{\partial f}{\partial R} \right)_{l,m} \right] = \kappa T_i^k, \quad (2)$$

where, as in (4),

$$f = (l^2 R)^{4/3}. \quad (3)$$

In equations (2), l is a characteristic length determining the value of the critical curvature $R \sim l^{-2}$, at which the nonlinear addition to the Lagrangian density L becomes substantial. As $l \rightarrow 0$, the equations pass into the Einstein equations. Relations (2), like the Einstein equations, contain the equations of motion for matter. This is easily verified by using the Bianchi identity and the rule for interchanging the indices of covariant differentiation

$$T_{i;k}^k = 0. \quad (4)$$

It is also easy to see that, for the chosen f in the form (3), all vacuum solutions of the Einstein equations (and, in particular, the Schwarzschild solution) satisfy the system (2).

Let us consider cosmological solutions of equations (2) with the interval given in the form

$$dS^2 = a^2(\eta) \left[d\eta^2 - d\chi^2 - \left(\frac{\sin \sqrt{\delta} \chi}{\sqrt{\delta}} \right)^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (5)$$

Here

$$a d\eta = c dt, \quad (6)$$

and the parameter δ is equal to 1, 0, -1 , respectively, for the closed, flat, and open models.

The nonzero components of the curvature tensor are

$$R_0^0 = \frac{3}{a^4} [(a')^2 - a''a], \quad R = -\frac{6(a\delta + a'')}{a^3},$$

$$R_\alpha^\beta = -\frac{1}{a^4} (2a^2\delta + a'^2 + aa'') \delta_\alpha^\beta, \quad (7)$$

where a prime denotes differentiation with respect to η . For the indicated values of the Ricci tensor, in system (2) there remain two independent equations with $i = k = 1$ and $i = k = 0$. As in solving the Einstein equations for an analogous problem, it is convenient, instead of the first of them, to use the following integral from (4):

$$\varkappa\varepsilon = 6a^{-3}l_1. \quad (8)$$

It is convenient further to introduce the following dimensionless quantities

$$b = \frac{a}{l_1}, \quad y = (b')^2 + \delta b^2. \quad (9)$$

Then formulas (7) take the form

$$l_1^2 R_0^0 = 3b^{-4} \left(y - \frac{b}{2} \frac{dy}{db} \right), \quad R l_1^2 = -\frac{3}{b^3} \frac{dy}{db}, \quad l_1^2 R_1^1 = -b^{-4} \left(y + \frac{b}{2} \frac{dy}{db} \right). \quad (10)$$

Relations (10), (8) allow equation (2), (3) for $i = k = 0$ to be written as follows:

$$y - \beta \left(\frac{dy}{db} \right)^{-2/3} \left[\frac{4}{9} \frac{d^2 y}{db^2} (y - \delta b^2) - \frac{1}{6} \left(\frac{dy}{db} \right)^2 + \frac{4}{3} \delta b \frac{dy}{db} \right] = 2b. \quad (11)$$

Here the notation has been introduced

$$\beta = \left(\frac{3l^2}{l_1^2} \right)^{1/3}. \quad (12)$$

The quantity l_1 , according to formula (8), determines the radius of the universe a at the moment when expansion stops in the closed model. The length l , as will be seen below, determines the minimum radius of the universe at which the change from contraction to expansion occurs. Since this is assumed to occur at very large curvatures and matter densities, the parameter β

Fig. 1

we shall regard as much less than unity.

Fig. 1

Figure 1: Fig. 1

Equation (11) has the exact solution

$$y = C(b - b_0), \quad (13)$$

where the quantities C and b_0 are determined by the positive roots of the equations

$$\begin{aligned} C - \frac{4}{3}C^{1/3}\beta\delta &= 2, \\ b_0 &= \beta C^{1/3}/6. \end{aligned} \quad (14)$$

For the analysis of the solutions it is convenient to make use of the smallness of β ; then from (14) we have

$$C \simeq 2, \quad b_0 \simeq (l/6l_1)^{2/3} \ll 1. \quad (15)$$

(For the model with a flat accompanying space ($\delta = 0$), these values are exact solutions of (14).)

Thus, for $b(\eta)$, according to formulas (6), (9), (13), (15), we have

$$(b')^2 = -\delta b^2 + 2b - 2b_0, \quad b d\eta = c dt/l_1. \quad (16)$$

These equations lead to the following parametric dependence of the radius of the universe b on the time t .

Closed model ($\delta = 1$):

$$\begin{aligned} b &= 1 - \sqrt{1 - 2b_0} \cos \eta, \\ ct/l_1 &= \eta - \sqrt{1 - 2b_0} \sin \eta. \end{aligned} \quad (17)$$

Flat model ($\delta = 0$):

$$\begin{aligned} b - b_0 &= \eta^2/2, \\ ct/l_1 &= \eta^3/6 + b_0\eta. \end{aligned} \quad (18)$$

Open model ($\delta = -1$):

$$\begin{aligned} b &= \sqrt{2b_0 + 1} \operatorname{ch} \eta - 1, \\ ct/l_1 &= \sqrt{2b_0 + 1} \operatorname{sh} \eta - \eta. \end{aligned} \quad (19)$$

The graph of the dependence $b(t)$ is given in Fig. 1.

For $b \gg b_0$, formulas (17)–(19) asymptotically go over into the corresponding Friedmann solutions ⁽⁵⁾. In the neighborhood of $t \sim 0$ (the change from contraction to expansion), the solutions behave in the same way:

$$b \simeq b_0 (1 + c^2 t^2 / 2l_1^2 b_0^2), \quad (20)$$

i.e., the radius of the universe passes through a regular minimum. The characteristic time t^* of the change from contraction to expansion, coinciding with the time when the deviation from the Einstein equations is substantial, according to formulas (12), (20), is, in order of magnitude, equal to

$$t^* \sim l^{2/3} l_1^{1/3} c^{-1}.$$

Taking account of the increase of entropy in the case of the closed model (17) leads, as shown in ⁽³⁶⁾, to an increase in the period of pulsations.

The author thanks A. G. Doroshkevich, Ya. B. Zeldovich, and A. A. Ruzmaikin for useful discussions.

Institute of Nuclear Physics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
18 V 1970

CITED LITERATURE

1. A. D. Sakharov, DAN, **177**, No. 1 (1967).
2. K. P. Stanyukovich, *Gravitation and Elementary Particles*, "Nauka," 1965, part 2, §§ 2, 4, 8.
3. Ya. B. Zeldovich, I. D. Novikov, *Relativistic Astrophysics*, "Nauka," 1967, a) Supplement 1; b) p. 20, § 5.
4. T. V. Ruzmaikina, A. A. Ruzmaikin, ZhETF, **57**, 681 (1969).
5. S. N. Breizman, V. P. Frolov, V. P. Sokolov, ZhETF, **7** (1970).
6. L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, "Nauka," 1967.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.