ON A METHOD FOR SOLVING PROBLEMS OF MATHEMATICAL PHYSICS

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Abstract

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Ya. N. FELD

ON A METHOD FOR SOLVING PROBLEMS OF MATHEMATICAL PHYSICS

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1. The idea of the method. The overwhelming majority of problems of mathematical physics reduce to finding a solution w (w may be a scalar, vector, matrix, etc.) satisfying certain N conditions. These conditions are such that the solution exists and is unique. One of these conditions is usually a differential equation, while the others have the character of initial or boundary conditions, conditions at infinity, and so on. As has already been said, the solution satisfying all these conditions is unique; however, if at least one of them is excluded, then there exists (usually) an infinite set of solutions satisfying the remaining conditions. This circumstance makes it possible to outline the following way of solving the problem. Let us exclude one* of the given N conditions, for example the n-th. Denote by w_n a solution satisfying the remaining $(N-1)$ conditions. There will be a set of such solutions; denote it by A_n ($w_n \in A_n$). If any other m-th ($m \neq n$) condition is excluded, then the solutions w_m satisfying the remaining $(N-1)$ conditions will constitute some new set A_m. The sets A_n and A_m are closed if the operators by means of which all N conditions of the problem are formulated are continuous. Obviously, both these sets have one common point, which is precisely the solution of the original problem. Consequently, it may be sought as the intersection of the indicated sets, $w = A_n \cap A_m$. The success of this approach depends on the ability to find general analytic expressions for the elements of the functional sets A_n and A_m. Usually they can be represented by expressions of the type $G_n\varphi$ and $G_m\psi$, where G_n and G_m are certain operators, and φ and ψ are functions belonging to the corresponding classes $\mathcal{L}_n$ and $\mathcal{L}_m$, i.e.

$$w_n = G_n\varphi, \quad \varphi \in \mathcal{L}_n; \qquad w_m = G_m\psi, \quad \psi \in \mathcal{L}_m. \tag{I}$$

When φ and ψ range over $\mathcal{L}_n$ and $\mathcal{L}_m$, w_n and w_m range over the sets A_n and A_m, respectively. From the existence of a unique common point of A_n and A_m it follows that there is a fixed pair φ and ψ for which the equality holds:

$$G_n\varphi = G_m\psi. \tag{II}$$

These functions also determine the common point, which is the desired solution of the original problem.

In a number of cases, for example when considering electrodynamic or acoustic problems, the analysis and exposition can be considerably simplified if one passes from the general solutions—the fields w, w_n, w_m (and their corresponding sets A_n, A_m)—to their asymptotic expressions as $R \to \infty$, valid in the far zone: $w \sim v,\ w_n \sim v_n,\ w_m \sim v_m$ ($A_n \sim a_n,\ A_m \sim a_m$). In this case the equalities (I) and (II) pass into the following:

$$v_n = g_n \varphi, \quad \varphi \in \mathcal{L}_n; \qquad v_m = g_m \psi, \quad \psi \in \mathcal{L}_m; \tag{Ia}$$

$$g_n \varphi = g_m \psi. \tag{IIa}$$

* One may also exclude two or more conditions.

Here g_n, g_m are the operators into which G_n, G_m pass in the far zone. The expression for the field v in the far zone is considerably simpler and, at the same time, it uniquely determines w in all space except for a certain region containing the sources.

The equality (II) or (IIa) can be used to find φ and ψ, after which the desired solution is determined by the expressions

$$w = G_n \varphi = G_m \psi; \qquad v = g_n \varphi = g_m \psi. \tag{III}$$

Fig. 1

Depending on the nature of the problem, equations (II) and (IIa) may be of different types and complexities and may be solved by different techniques. However, one can indicate a general method of solution based on the fact that w (v) is the only common point of two sets A_n and A_m (a_n and a_m). Introducing some metric space M containing both these sets, it is easy to construct* an iterative process leading to the desired solution. To do this, taking an arbitrary element $w_n^0 \in A_n$, we find the element $w_m^{(1)} \in A_m$ nearest to it; then we find the element $w_n^{(2)} \in A_n$ nearest to the latter, and so on. If this process converges, then

$$w = \lim_{\nu \to \infty} w_n^{(2\nu)} = \lim_{\nu \to \infty} w_m^{(2\nu+1)}, \tag{IV}$$

and v is found analogously.

Let us illustrate these general arguments with concrete examples.

2. Diffraction of an electromagnetic wave by a metallic screen. Let a primary wave $\mathbf{E}^0, \mathbf{H}^0$ be incident on a metallic ideally conducting screen S (Fig.

1); we denote the secondary—diffracted—field by $\mathbf{E}, \mathbf{H}$. It is uniquely determined by the following conditions: 1) $\mathbf{E}, \mathbf{H}$ is the field produced by currents distributed on S; 2) $\mathbf{E}, \mathbf{H}$ is a field having no sources outside the surface $S+\Sigma$ and satisfying, on the outer side of S, the equality $[\mathbf{n}(\mathbf{E}+\mathbf{E}^0)] = 0$. Here Σ is a surface completing S to a closed one (Fig. 1); in particular, it may close at infinity, and $\mathbf{n}$ is the normal to $S+\Sigma$. In the present case $N=2$. The set A_1 consists of all fields satisfying condition 2, and A_2 of fields satisfying condition 1. As M, it is convenient to consider the linear space whose elements are electromagnetic fields $w = \{\mathbf{E}, \mathbf{H}\}$, the sources of which are located inside the surface $S+\Sigma$ or on it. We introduce in M the norm

$$\|w\| = \left(\operatorname{Re}\int_{S_0} [\mathbf{E}\mathbf{H}^*]\, d\mathbf{s}\right)^{1/2}. \tag{1}$$

Here S_0 is an arbitrary surface enclosing $S+\Sigma$.

Then $\|w_1 - w_2\|$ is the distance between the elements w_1 and w_2 of the space M.

Let us further specialize the problem. We shall regard the screen S as part of an infinite cylinder along the generator z. If the field $\mathbf{E}^0, \mathbf{H}^0$ does not depend on the coordinate z, and the vector $\mathbf{E}^0$ is polarized parallel to the z-axis, then the problem reduces to a plane scalar problem with Dirichlet boundary conditions on S, where S should now be understood as the arc formed by the intersection of the screen with the plane $z = \text{const}$. The initial relations in this case are as follows:

$$E = E_z; \qquad (\nabla^2 + k^2)E = 0; \qquad E = -E^0 \text{ on } S. \tag{2}$$

For E, the radiation conditions and the Meixner conditions at the edges of the screen must also be satisfied. Since the problem is formulated (see (2)) only in terms of the component $E_z = E$, it is convenient henceforth to put $w \equiv E$, retaining formula (1) for $\|w\|$.

* An analogous process is used in work (1), where the nearest elements of two nonintersecting sets are found.

The elements of the sets A_1 and A_2 can now be written in the form:

$$w_1 = \int_S E^0 \frac{\partial G}{\partial n}\, ds - \int_\Sigma \varphi \frac{\partial G}{\partial n}\, ds, \tag{3}$$

$$w_2 = \int_S \psi H_0^{(2)}(kr)\, ds, \tag{4}$$

where G is the Green' s function of the Helmholtz equation for the domain V_e, exterior to $S+\Sigma$, vanishing on $S+\Sigma$; formula (3) is valid only for $\overline{V_e}$; r is the distance between the points of integration and observation; φ and ψ are arbitrary* functions having the following physical meaning: φ is the value of E on Σ, and ψ is the current** on S; ds is an element of arc. Equation (II) for the problem under consideration has the form

$$\int_S \psi H_0^{(2)}(kr)\, ds + \int_\Sigma \varphi \frac{\partial G}{\partial n}\, ds = \int_S E^0 \frac{\partial G}{\partial n}\, ds \tag{5}$$

and must be satisfied for every point of V_e.

Let us test the effectiveness of the proposed method on a simple example, when S is the segment of the straight line $b_- \leq x \leq b_+$ (a strip), while Σ completes it to the whole x-axis; V_e is a half-plane. The Green' s function then has the form

$$G(p,q) = \frac{1}{4i}\{H_0^{(2)}(kr_{pq}) - H_0^{(2)}(kr_{pq^*})\}, \tag{6}$$

where q and q^* are points mirror-symmetric with respect to the x-axis, and r_{pq} (r_{pq^*}) is the distance between the points p and q (p and q^*). Equality (5) will now be written as follows:

$$\int_\Sigma \varphi(p) H_1^{(2)}(kr_{pq}) \sin\alpha\, dx_p - \frac{2i}{k}\int_S \psi(p) H_0^{(2)}(kr_{pq})\, dx_p =$$

$$= \int_S E^0(p) H_1^{(2)}(kr_{pq}) \sin\alpha\, dx_p, \qquad (q \in V_e), \tag{7}$$

where α is the angle between $\mathbf{r}_{pq}$ and the x-axis. Placing the observation point q in the far zone and replacing $H_0^{(2)}$, $H_1^{(2)}$, r_{pq}, and α by their asymptotic expressions, we obtain, instead of (7), an equation of type (IIa)

$$\sin\beta \int_\Sigma \varphi(x) e^{ikx\cos\beta}\, dx - \frac{2}{k}\int_S \psi(x) e^{ikx\cos\beta}\, dx = \sin\beta \int_S E^0(x) e^{ikx\cos\beta}\, dx. \tag{8}$$

Here β is the angle between the radius vector drawn from the origin to the observation point and the x-axis. Introducing the analytic functions

$$F(u) = 2\int_{b_-}^{b_+} \psi(x) e^{iux}\, dx; \qquad f(u) = \int_{b_-}^{b_+} E^0(x) e^{iux}\, dx,$$

$$\Phi_+(u) = \int_{b_+}^{\infty} \varphi(x) e^{iux}\, dx, \quad \operatorname{Im} u \geq 0; \qquad \Phi_-(u) = \int_{-\infty}^{b_-} \varphi(x) e^{iux}\, dx, \quad \operatorname{Im} u \leq 0, \tag{9}$$

where F and f are entire for $|b_+| < \infty$ and $|b_-| < \infty$, while Φ_+ and Φ_- are holomorphic in the corresponding half-planes, we reduce (8) (using the notation $u = k\cos\beta$) to the functional equation

$$-F(u) + \sqrt{k^2 - u^2}(\Phi_+(u) + \Phi_-(u)) = \sqrt{k^2 - u^2} f(u). \tag{10}$$

Owing to the analyticity of all expressions entering into (10), it must be satisfied throughout the entire u-plane. This equation can be solved exactly, for example, in the case when $b_- = 0$, $b_+ = \infty$ (a half-plane). Indeed, then $\Phi_+(u) = 0$, while $F(u) = F_+(u)$ and $f(u) = f_+(u)$, i.e., they turn out to be holomorphic only in the upper half-plane. Equality (10)

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* They must be sufficiently smooth, and ψ may have a singularity at the ends of the arc S allowed by Meixner' s conditions.

** ψ differs from the current by a constant factor.

takes the form (an inhomogeneous Hilbert problem)

$$-F_+(u) + \sqrt{k^2 - u^2}\, \Phi_-(u) = \sqrt{k^2 - u^2}\, f(u). \tag{11}$$

This equation is solved elementarily by the known method [2]. Temporarily assuming $\operatorname{Im} k < 0$, we obtain

$$\Phi_-(u) = \frac{i}{2\pi\sqrt{k+u}} \int_{-\infty}^{\infty} f(t)\, \frac{\sqrt{k+t}}{t-u}\, dt, \qquad \operatorname{Im} u < 0. \tag{12}$$

To compute this integral, one must specify the field E^0. If $E^0 = \exp[-ikR\cos(\beta - \beta_0)]$ is a plane wave and $\cos\beta_0 > 0$, then (see (9))

$$f(t) = i(t - k\cos\beta_0)^{-1}. \tag{13}$$

Substituting this expression into (12), we obtain

$$\Phi_-(u) = \frac{1}{i(k_1 - u)} + i\sqrt{k + k_1}/\sqrt{k+u}\,(k_1 - u); \qquad k_1 = k\cos\beta_0. \tag{14}$$

Knowing Φ_- and f, with the aid of (11) we find the directivity diagram $F_+(u)$ $(u = k\cos\beta)$

$$F \equiv F_+ = (2i\sin\beta/2\cos\beta_0/2)/(\cos\beta_0 - \cos\beta).$$

It corresponds to a cylindrical wave emanating from the edge of the half-plane. To obtain the current $\psi(x)$, it is sufficient to take the Fourier transform of $F(u)$. In the case of a finite strip $(b = b_+ = -b_-)$, equation (10) can be solved using the iterative process indicated above. Generalizing it somewhat, we introduce two norms *

$$\|v\|_1 = \left(\int_{-\infty}^{\infty} \left|v(u) - v(0) - c_+(e^{ibu} - 1) - c_-(e^{-ibu} - 1)\right|^2 \frac{du}{u^2}\right)^{\frac{1}{2}};$$

$$\|v\|_2 = \left(\int_{-\infty}^{\infty} \left|\frac{v(u)}{\sqrt{k^2 - u^2}}\right|^2 du\right)^{\frac{1}{2}}, \tag{15}$$

where

$$c_+ = \mathcal{F}_{b-0}\frac{v(u) - v(0)}{iu}; \qquad c_- = \mathcal{F}_{-b+0}\frac{v(u) - v(0)}{iu}; \qquad \mathcal{F}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{-iux}.$$

We shall use the first of these to determine $v_2 \in a_2$, closest to the given v_1, and the second to find $v_1 \in a_1$, closest to the given v_2. The elements a_2 and a_1 are now the functions (see (9))

$$v_2 = F(u), \qquad v_1 = \sqrt{k^2 - u^2}\{\Phi_+(u) + \Phi_-(u) - f(u)\}. \tag{16}$$

The iterative process is determined by the conditions

$$\|v_2^{(2\nu)} - v_1^{(2\nu-1)}\|_1 = \min; \qquad \|v_1^{(2\nu+1)} - v_2^{(2\nu)}\|_2 = \min.$$

Hence, taking into account (15), (16), (9), the recurrence formulas follow:

$$v_2^{(2\nu)}(u) = v_2^{(2\nu)}(0) + u\int_{-b}^{b} e^{iux}\mathcal{F}_x\{[v_1^{(2\nu-1)}(\xi) - v_1^{(2\nu-1)}(0)]\xi^{-1}\}\, dx -$$

$$-c_+^{(2\nu-1)}(e^{ibu} - 1) - c_-^{(2\nu-1)}(e^{-ibu} - 1),$$

$$v_1^{(2\nu+1)}(u) = \sqrt{k^2-u^2}\left[\left(\int_{-\infty}^{-b} + \int_{b}^{\infty}\right) e^{iux}\mathcal{F}_x\left\{\frac{v_2^{(2\nu)}(\xi)}{\sqrt{k^2-\xi^2}}\right\} dx - f(u)\right].$$

Here

$$\mathcal{F}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi x};$$

$v_2^{(2\nu)}(0)$ is found from the condition $v_2^{(2\nu)}(u) \to 0$ as $u \to \pm\infty$ along the real axis.

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CITED LITERATURE

[1] D. E. Vakman, *Regular Method for the Synthesis of FM Signals*, Moscow, 1967.
[2] N. I. Muskhelishvili, *Singular Integral Equations*, 1946.

$$^{*}\ [(v(u) - v(0))u^{-1} \in L_2[-\infty, \infty], \qquad \operatorname{Im} k < 0.$$

Note: Figure translations are in progress. See original paper for figures.