ON THE ZEROS OF PERIODIC DIRICHLET FUNCTIONS

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Abstract

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ON THE ZEROS OF PERIODIC DIRICHLET FUNCTIONS

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The idea of shortened functional equations from ([1,2]), which forms the basis of the theory of various kinds of estimates, uniform in all parameters, for Dirichlet functions (see ([3,4])), can also be used to study the location of the zeros of some of these functions. Along these lines, in particular, one obtains a detailed picture of the analytic aspect of the problem of zeros of periodic Dirichlet functions.

In the present article we confine ourselves to stating results for the two simplest classes of periodic Dirichlet functions.

Namely, we consider functions $Z(s)$, defined for $\operatorname{Re} s > 1$ by Dirichlet series

$$Z(s) = \sum_{n=0}^{\infty} a_n \lambda_n^{-s} \quad (1 = \lambda_0 < \lambda_1 < \cdots) \tag{1}$$

with real coefficients $a_0 \neq 0, a_1, a_2, \ldots$; meromorphic, with a simple pole at the point $s = 1$; having period $2\pi i / \ln q$ with arbitrary $q > 1$, and the functional equation

$$q^s Z(s) = q^{1-s} Z(1-s). \tag{2}$$

In particular, when q is equal to a power of a prime number, and under special additional restrictions on the sequence of coefficients $a_0, a_1, a_2, \ldots$, the series (1) $Z(s)$ are zeta-functions of fields (of the second kind) of algebraic functions over a finite field of constants ([5,6]).

With the aid of the shortened functional equation corresponding to equation (2) (the case of equation (1) from ([7]) with $\varphi(s) = \psi(s) = Z(s)$, $\alpha_\nu = 0$, $\beta_\nu = 1$, $\lambda = 1$, $A = B = q$), we obtain:

1°. For every fixed value $q > 1$, the described class of functions $Z(s)$ consists of a countable set of representatives.

2°. $\operatorname*{res}_{s=1} Z(s) = (a_2 - a_0 q)/q(q-1)\ln q$, where $a_0,\ a_2$ are coefficients of the series (1).

3°. In the half-plane $\operatorname{Re} s \geqslant 1/2$, the zeros of $Z(s)$ lie on one or on two vertical straight lines.

4°. The location of the zeros of $Z(s)$ is determined by the simplest relations between the parameter q and the first three coefficients a_0, a_1, a_2 of its Dirichlet series (1).

Theorem 1. Let

$$A = q + 1 - a_1/a_0, \qquad C = A^2 - 4(a_2/a_0 - q - (q+1)a_1/a_0),$$

$$B = \sqrt{|C|}.$$

I. If $C \geqslant 0$, then in the half-plane $\operatorname{Re} s \geqslant 1/2$ we have:

1°. If

$$|A - B| \leq 4\sqrt{q}, \qquad |A + B| \leq 4\sqrt{q}, \tag{3}$$

then all zeros of $Z(s)$ lie on the line $\operatorname{Re} s = 1/2$, and their ordinates t_1, t_2 are computed by the formula

$$\cos t_\nu \ln q = [A + (-1)^\nu B]/4\sqrt{q} \quad (\nu = 1, 2). \tag{4}$$

2°. If

$$|A - B| \leq 4\sqrt{q}, \qquad |A + B| > 4\sqrt{q}, \tag{5}$$

then $Z(s)$ has zeros on the line $\operatorname{Re} s = 1/2$ with ordinates t determined from the equality

$$\cos t \ln q = (A - B)/4\sqrt{q}, \tag{6}$$

and zeros on the line $\operatorname{Re} s = \sigma > 1/2$, determined by the equation

$$q^\sigma + q^{1-\sigma} = A + B \quad \text{or} \quad q^\sigma + q^{1-\sigma} = -(A + B), \tag{7}$$

depending on whether $A + B > 0$ or $A + B < 0$, with ordinates t, respectively equal to

$$2\pi k/\ln q \quad \text{or} \quad \pi(2k+1)/\ln q \quad (k = 0, \pm 1, \ldots). \tag{8}$$

3°. If

$$|A - B| > 4\sqrt{q}, \qquad |A + B| \leq 4\sqrt{q}, \tag{9}$$

then $Z(s)$ has zeros on the line $\operatorname{Re} s = 1/2$ with ordinates t determined from the equality

$$\cos t \ln q = (A + B)/4\sqrt{q}, \tag{10}$$

and zeros on the line $\operatorname{Re} s = \sigma > 1/2$, where σ is a root of the equation

$$q^{\sigma} + q^{1-\sigma} = A - B \quad \text{or} \quad q^{\sigma} + q^{1-\sigma} = B - A \tag{11}$$

depending on whether $A > B$ or $A < B$, with ordinates t, respectively equal to

$$2\pi k/\ln q \quad \text{or} \quad \pi(2k+1)/\ln q \quad (k = 0, \pm 1, \ldots). \tag{12}$$

4°. If

$$|A - B| > 4\sqrt{q}, \qquad |A + B| > 4\sqrt{q}, \qquad A \neq 0, \tag{13}$$

then the zeros of $Z(s)$ lie on the vertical lines $\operatorname{Re} s = \sigma_1 > 1/2$ and $\operatorname{Re} s = \sigma_2 > 1/2$, where σ_1 and σ_2 are roots of the equations:

$$q^{\sigma_1} + q^{1-\sigma_1} = \pm(A - B) \quad \text{and} \quad q^{\sigma_2} + q^{1-\sigma_2} = \pm(A + B), \tag{14}$$

and the ordinates, depending on whether

$$A - B > 0 \quad \text{or} \quad A - B < 0, \quad \text{and} \quad A + B > 0 \quad \text{or} \quad A + B < 0, \tag{15}$$

will be respectively equal to

$$2\pi k/\ln q \quad \text{or} \quad \pi(2k+1)/\ln q \quad (k = 0, \pm 1, \ldots). \tag{16}$$

5°. If

$$|A - B| > 4\sqrt{q}, \qquad |A + B| > 4\sqrt{q}, \qquad A = 0, \tag{17}$$

then all zeros of $Z(s)$ lie on one line $\operatorname{Re} s = \sigma > 1/2$, where σ is a root of the equation

$$q^{\sigma} + q^{1-\sigma} = B, \tag{18}$$

and the ordinates of the zeros are equal to

$$\pi k/\ln q \quad (k = 0, \pm 1, \ldots). \tag{19}$$

II. For $C < 0$ in the half-plane $\operatorname{Re} s \geq 1/2$ we have: all zeros of $Z(s)$ lie in the half-plane $\operatorname{Re} s = \sigma > 1/2$, and their abscissa σ and ordinates t are the roots of the following system of equations:

$$2(q^{\sigma} + q^{1-\sigma}) \cos t \ln q = A, \qquad 2(q^{\sigma} - q^{1-\sigma}) \sin t \ln q = B.$$

III. All the situations listed above for the disposition of the zeros of the functions $Z(s)$ do in fact occur and are the only possible ones.

IV. There exists a method that makes it possible to construct functions $Z(s)$ realizing any of the above-indicated situations for the location of zeros.

Converse theorem. 1°. If all zeros of $Z(s)$ lie on the line $\operatorname{Re} s = 1/2$, then the relations (3), (4) and $C \geq 0$ hold.

2°. If $Z(s)$ has zeros on the line $\operatorname{Re} s = 1/2$ and on the line $\operatorname{Re} s = \sigma > 1/2$, then the assertions in the form (5)–(8) or (9)–(12) hold, depending on whether

$$|A + B| > |A - B| \quad \text{or} \quad |A + B| < |A - B| \quad \text{and } C > 0.$$

3°. If the zeros of $Z(s)$ lie on two different lines situated to the right of $\operatorname{Re} s > 1/2$, then the assertions in the form (13)–(16) hold.

4°. If the zeros of $Z(s)$ lie on one line $\operatorname{Re} s = \sigma > 1/2$, then the assertions in the form (17)–(19) hold.

Corollary. An analogue of the Riemann hypothesis for zeros holds for precisely those functions $Z(s)$ for which the first three coefficients of their Dirichlet series (1) and the parameter q satisfy the inequalities

$$C \geq 0, \qquad |A - B| \leq 4\sqrt{q}, \qquad |A + B| \leq 4\sqrt{q}.$$

If $C \geq 0$, but at least one of the last two inequalities is not satisfied, then the corresponding function $Z(s)$ has a real (“Siegel”) zero, or a zero with ordinate $t = \pi/\ln q$, lying to the right of the line $\operatorname{Re} s = 1/2$.

Let, further, $Z^{*}(s)$ be a Dirichlet function differing from $Z(s)$ in that, instead of the functional equation (2), it satisfies the equation

$$Z^*(s) = Z^*(1-s).$$

This class of Dirichlet functions likewise consists of an uncountable set of representatives for any fixed $q > 1$. In particular, it includes zeta-functions of elliptic function fields ([8],[9]). Concerning the location of the zeros of $Z^*(s)$, the following holds.

Theorem 2. 1°. All zeros of the function $Z^*(s)$ lie on the line $\operatorname{Re} s = 1/2$ if and only if the first two coefficients of its Dirichlet series (1) satisfy the inequality

$$|a_1/a_0 - q - 1| \le 2\sqrt{q}, \tag{20}$$

and the ordinates of the zeros t are computed by the formula

$$\cos t \ln q = [a_1 - a_0(q+1)]/2a_0\sqrt{q}.$$

2°. If inequality (20) is not satisfied, then the function $Z^*(s)$ has a real ("Siegel") zero or a zero with ordinate $t = \pi/\ln q$, lying in the half-plane $\operatorname{Re} s > 1/2$.

3°. One can actually indicate arbitrarily many functions $Z^*(s)$ for which assertion 1° is valid and, with any $1/2 < \operatorname{Re} s < 1$, assertion 2° is valid for every $q > 1$.

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Note: Figure translations are in progress. See original paper for figures.