ON THE S-NUMBERS OF POSITIVE OPERATORS

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Abstract

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P. E. SOBOLEVSKII

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The paper is devoted to estimates of the S-numbers (see, for example, ([1])) of certain functions of positive operators. For them estimates are established which coincide in order with the corresponding estimates for self-adjoint operators.

1. A linear operator A, acting in a space E, with dense domain of definition, is weakly positive (w. p. E-operator) if

$$\|(A+tI)^{-1}\| \leq c(1+t)^{-1} \quad (t \geq 0).$$

If, however,

$$\|(A+tI)^{-1}\| \leq c(1+|t|)^{-1}$$

for $\operatorname{Re} t \geq 0$, then A is strongly positive (s. p. E-operator). If A is a w. p. E-operator, then all its powers A^{α} are defined, and $A^{-\alpha}$ for $\alpha \geq 0$ is an analytic semigroup, and all $A^{-\alpha}$ for $\alpha > 0$ are completely continuous if $A^{-\alpha_0}$ is completely continuous for some $\alpha_0 > 0$. If, however, A is an s. p. E-operator, then it generates an analytic semigroup $\exp\{-tA\}$ with exponentially decreasing norm, and for small $t > 0$ the estimate

$$\|[\exp\{-tA\}]'\| \leq ct^{-1}$$

holds. The generator of the semigroup $A^{-\alpha}$ $(\alpha \geq 0)$ is naturally denoted by $\ln A$. We shall assume that it is an s. p. E-operator. This is always true for the operator mA for sufficiently large $m > 0$. Then the operators $(\ln A)^{\alpha}$ are defined. The generator of the semigroup $(\ln A)^{-\alpha}$ $(\alpha \geq 0)$ is denoted by $-\ln\ln A$, etc. (see ([2]) and the bibliography there).

2. **Lemma 1.** *The inequalities*

$$\sum_{i=1}^{m} s_i(A^{-\alpha}) \leq c(\alpha_0-\alpha)^{-1} \sum_{i=1}^{m} s_i^{\alpha/\alpha_0}(A^{-\alpha_0}) \quad (0 \leq \alpha < \alpha_0;\ m = 1, 2, \ldots). \tag{1}$$

In the proof one uses the inequalities for the s-numbers of the sum of completely continuous operators (see ([1])) and the integral representation of fractional powers (see ([2])).

Following ([1]), we shall call a sequence $a_i \geq 0$ regular if

$$\sum_{i=1}^{m} a_i \leq Rma_m \quad (0 \leq R < \infty,\ m = 1, 2, ...). \tag{2}$$

If $a_i = a_i(\delta)$ is regular for every δ in $[0, \delta_0]$, $\delta_0 > 0$, and $R(\delta) \leq \overline{R} < \infty$, then we shall say that the sequence a_i is uniformly regular in δ.

From Lemma 1 and the inequalities for the s-numbers of the product of completely continuous operators (see ([1])) it follows that

Theorem 1. *Let the sequence $a_i = S_i^{\delta}(A^{-\alpha_0})$ be uniformly regular in δ. Then the inequalities*

$$s_m(A^{-\alpha}) \leq c(\bar{\alpha}) s_m^{\alpha/\alpha_0}(A^{-\alpha_0}) \quad (0 \leq \alpha \leq \bar{\alpha},\ m = 1, 2, ...). \tag{3}$$

Remark 1. Important examples of positive operators whose s-numbers are regular are operators in the space L_2 generated by various boundary-value problems for semibounded elliptic operators.

Lemma 2. *Let F be a linear closed operator with $D(F) \supset D(A^{\gamma})$ for some $\gamma \geq 0$, and*

$$\|Fx\| \leq c\|A^{\gamma}x\|^{\lambda}\|x\|^{1-\lambda} \quad (x \in D(A^{\gamma})). \tag{4}$$

for some $0 \leq \lambda \leq 1$. Let, finally, $\lambda\gamma < \alpha < \alpha_0 + \lambda\gamma$. Then

$$\sum_{i=1}^{m} s_i(FA^{-\alpha}) \leq c_1(\alpha-\lambda\gamma)^{-1}(\alpha_0+\lambda\gamma-\alpha)^{-1} \sum_{i=1}^{m} s_i^{(\alpha-\lambda\gamma)/\alpha_0}(A^{-\alpha_0}) \quad (m = 1, 2, ...). \tag{5}$$

Theorem 2. Let the conditions of Theorem 1 be satisfied, and let the operator F be as in Lemma 2. Then

$$s_m(FA^{-\alpha}) \leq c(\bar{\alpha})(\alpha - \lambda\gamma)^{-1} s_m^{(\alpha-\lambda\gamma)/\alpha_0}(A^{-\alpha_0}) \quad (\lambda\gamma < \alpha \leq \bar{\alpha}, \quad m = 1, 2, ...). \tag{6}$$

Remark 2. If $FA^{-\lambda\gamma}$ is bounded, then $s_m(FA^{-\alpha}) \leq \|FA^{-\lambda\gamma}\| \cdot \|s_m(A^{-\alpha+\lambda\gamma})\|$, and (6) follows from (3). As is known, (4) may also be valid when $FA^{-\lambda\gamma}$ is unbounded. Nevertheless, (6) still holds.

3. Theorem 3. If $(I - A^{-1})^{-1}$ is bounded, then $(\ln A)^{-1}$ is bounded and

$$(\ln A)^{-1} = (I - A^{-1})^{-1} \int_0^\infty \frac{t-1}{t} \frac{1}{\pi^2 + \ln^2 t} (A + tI)^{-1} \, dt. \tag{7}$$

If, however, $(I - zA^{-1})^{-1}$ is bounded for all $0 \leq z \leq 1$, then $\ln A$ is a c.n. E-operator, and for $(\ln A + \lambda I)^{-1}$ $(\lambda \geq 0)$ an integral representation is valid which is obtained from (7) by replacing A by $e^\lambda A$.

This theorem makes it possible to estimate the s-numbers of the logarithm of a positive operator in terms of the s-numbers of the operator itself. For simplicity of formulation we put $\alpha_0 = 1$.

Lemma 3. Let $(I - A^{-1})^{-1}$ be bounded. Then

$$\sum_{i=1}^m s_i([\ln A]^{-1}) \leq c \sum_{i=1}^m [D - \ln s_i(A^{-1})]^{-1} \quad (D > \ln s_1(A^{-1}), \quad m = 1, 2, \ldots). \tag{8}$$

Let $(I + zA^{-1})^{-1}$ be bounded for all $0 \leq z \leq 1$. Then

$$\sum_{i=1}^m s_i([\ln A]^{-\alpha}) \leq c_1 \sum_{i=1}^m [D - \ln s_i(A^{-1})]^{-\alpha}$$

$$(D > \ln s_1(A^{-1}), \quad 0 \leq \alpha \leq 1, \quad m = 1, 2, \ldots). \tag{9}$$

Theorem 4. Let $(I - A^{-1})^{-1}$ be bounded and let the sequence $a_i = [D - \ln s_i(A^{-1})]^{-1}$ be regular. Then

$$s_m([\ln A]^{-1}) \leq \tilde{c} \, [D - \ln s_m(A^{-1})]^{-1} \quad (m = 1, 2, \ldots). \tag{10}$$

Let $(I - zA)^{-1}$ be bounded for all $0 \leq z \leq 1$ and let the sequence $a_i = [D - \ln s_i(A^{-1})]^{-\delta}$ be uniformly regular in δ. Then

$$s_m([\ln A]^{-\alpha}) \leq c(\bar{\alpha})[D - \ln s_i(A^{-1})]^{-\alpha} \quad 0 \leq \alpha \leq \bar{\alpha}, \quad m = 1, 2, \ldots. \tag{11}$$

Remark 3. Analogously one estimates the s-numbers of $(\ln \ln A)^{-\alpha}$, etc. The theorem below solves, in a certain sense, the inverse problem: from the s-numbers of $(\ln A)^{-1}$ the s-numbers of $A^{-\alpha}$ are estimated.

Theorem 5. Let A be a c.n. E-operator, and let the sequence $a_i = s_i(A^{-1})$ be regular. Then

$$\ln s_m(\exp\{-tA\}) \leq c - Dt \, s_m^{-1}(A^{-1}) \quad (0 \leq t \leq t_0, \quad m = 1, 2, \ldots). \tag{12}$$

In the proof, estimates of derivatives of semigroups are used (see § 1).

4. Here estimates of s-numbers in the norms of the classes σ_p are given (see (1)). In this case the regularity conditions are dropped.

Theorem 6. Let $A^{-\alpha_0} \in \sigma_{p_0}$ $(\alpha_0 > 0,\ p_0 > 0)$. Then

$$\|A^{-\alpha}\|_{\sigma_{p_0\alpha/\alpha_0}} \leq c(\bar{\alpha})\|A^{-\alpha_0}\|^{\alpha/\alpha_0}_{\sigma_{p_0}} \quad (0 \leq \alpha \leq \bar{\alpha}). \tag{13}$$

If the operator F is the same as in Lemma 2, then

$$\|FA^{-\alpha}\|_{\sigma_{p_0(\alpha-\lambda\gamma)/\alpha_0}} \leq c(\bar{a})(\alpha - \lambda\gamma)^{-1}\|A^{\alpha_0}\|_{\sigma_{p_0}} \qquad (\lambda\gamma < \alpha \leq \bar{a}). \tag{14}$$

Remark 4. Similar assertions are also valid for $s_m([\ln A]^{-\alpha})$, etc.

Remark 5. In the proof of Theorem 6 only the boundedness in l_p $(p > 1)$ of the Hardy operator is used. Therefore estimates (13) and (14) also hold for other classes of completely continuous operators in which this operator is bounded.

5. Here, in the norms of symmetrically normed ideals σ_Φ (see (1)), the s-numbers of semigroups generated by S.P. E-operators A and B with $D(A) = D(B)$ are compared.

Theorem 7. Let

$$\|\exp\{-tA\}\|_{\sigma_\Phi} \leq \varphi(1/t) \qquad (0 < t \leq 1),$$

where $\varphi(u)$ is a continuous function on $[1, +\infty)$ satisfying the condition

$$\varphi(u) \leq c(\alpha)\varphi(\alpha u) \qquad (0 < \alpha \leq 1). \tag{15}$$

Then

$$\|\exp\{-tB\}\|_{\sigma_\Phi} \leq c\varphi(1/t) \qquad (0 < t \leq 1).$$

Remark 6. The conditions of Theorem 7 are satisfied by elliptic operators.

If, however, conditions (15) are not fulfilled, one can use the following proposition, which, in our opinion, is also of independent interest.

Theorem 8. Let $\varphi(u)$ and $\psi(u)$ be absolutely continuous, monotonically increasing positive functions on $[1, +\infty)$, satisfying the conditions $\varphi(1) = 1$ and

$$[\ln\psi(u)]' \leq c_1[\ln\varphi(u)]'. \tag{16}$$

Let $b(v)$ be a continuous decreasing positive function on $[1, +\infty)$, satisfying the condition

$$b[\varphi(u)]\varphi'(u) \leq c_2\psi'(u). \tag{17}$$

Finally, let $a_i(u)$ be a sequence of nonnegative functions on $[1, +\infty)$, bounded for each u, and satisfying the condition

$$\sum_{i=1}^{\infty} a_i(u) \leq \varphi(u). \tag{18}$$

Then on $[1, +\infty)$ the inequality

$$\sum_{i=1}^{\infty} b(i)a_i(u) \leq c_2\left[c_1 + \sup_i a_i(u)\right]\psi(u) \tag{19}$$

is valid.

Remark 7. The same assertion also holds for integrals. Now let (see (1)) $\pi_1 = 1$, $\pi_{i+1} \leq \pi_i$, $i = 1, 2, ...$, and

$$\|\exp\{-tA\}\|_{\sigma_\pi} \overset{\text{def}}{=} \sum_{i=1}^{\infty} \pi_i s_i(\exp\{-tA\}) \leq \varphi(1/t) \qquad (0 < t \leq 1), \tag{20}$$

but the function $\varphi(u)$ does not satisfy (15). Theorem 8 makes it possible to construct a sequence π_i' such that $\|\exp\{-tA\}\|_{\sigma_{\pi'}} \leq \psi(1/t)$, and $\psi(u)$ already satisfies (15). Then, by Theorem 6,

$$\|\exp\{-tB\}\|_{\sigma_{\pi'}} \leq c\psi(1/t).$$

For example, if $\varphi(u) = \exp u$, then one can put $\psi(u) = u$ and $\pi_i' = \frac{1}{i}\pi_i$.

Voronezh State University
named after the Lenin Komsomol

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REFERENCES

1. I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, "Nauka," 1965.

2. M. A. Krasnosel' skii, P. P. Zabreiko et al., *Integral Operators in Spaces of Summable Functions*, "Nauka," 1966.

Note: Figure translations are in progress. See original paper for figures.