SOME SIMPLE EXAMPLES OF UNIVERSAL FUNCTIONS

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Abstract

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MATHEMATICS

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For any positive natural number m, consider the polynomial

$$v_m(x, y) = \frac{1}{2} m(x + y)(x + y + 1) + x - (m - 1)y.$$

By direct computation one verifies the equalities

$$\begin{aligned} v_m(0, 0) &= 0, \\ v_m(0, x + 1) &= v_m(x, 0) + 1, \\ v_m(x + 1, y) &= v_m(x, y + 1) + m. \end{aligned}$$

With the help of these equalities it is easy to see that at points with natural-number coordinates the polynomial v_m takes natural values, and at distinct points—distinct ones.

Theorem 1. Let m be a positive natural number and let ω be a (partial) arithmetical function of two variables such that, for any natural x, y, z, the equalities

$$\omega(v_m(y, 0), z) \simeq v_m(y, z), \tag{1}$$

$$\omega(v_m(0, x + 1), z) \simeq x, \tag{2}$$

$$\omega(v_m(y + 1, x + 1), z) \simeq \omega(\omega(x, z), \omega(y, z)). \tag{3}$$

hold.

Under these assumptions ω is a recursively complete arithmetical operation in the sense of (3), i.e., there exist natural numbers P, Q, R, and S such that the following 4 conditions are satisfied for any natural x, y, z:

a) $\omega(P, x) \simeq x + 1$;

b) $\omega(Q, x+1) \simeq x$;

c)

$$\omega(\omega(\omega(R, x), y), z) \simeq \begin{cases} y, & \text{if } x = 0, \\ z, & \text{if } x \neq 0; \end{cases}$$

d) $\omega(\omega(S, x), y)$ is defined and

$$\omega(\omega(\omega(S, x), y), z) \simeq \omega(\omega(x, z), \omega(y, z)).$$

On the basis of the results proved in (^{3}), it follows from this that

Theorem 2. Under the assumptions of Theorem 1 one may assert:

a) if φ is a one-place arithmetical function that is μ-recursive relative to ω, then there exists a natural number p such that for every natural z the equality

$$\varphi(z) \simeq \omega(p, z);$$

holds;

b) if n is a positive natural number and φ is an $(n+1)$-place arithmetical function that is μ-recursive relative to ω, then there exists a natural number p such that, for any natural $z_1, z_2, \dots, z_n, z_{n+1}$, the equality

$$\varphi(z_1, z_2, \dots, z_n, z_{n+1}) \simeq \omega(\omega(\dots\omega(\omega(p, z_1), z_2), \dots, z_n), z_{n+1}),$$

moreover the expression $\omega(\dots\omega(\omega(p, z_1), z_2), \dots, z_n)$ is meaningful for any natural $z_1, z_2, \dots, z_n$.

Corollary. Under the hypotheses of Theorem 1 and under the additional assumption that the function ω is partially recursive, one may assert that ω is a principal universal function (in the sense of (4)) for partially recursive functions of one argument.

Using the recursion theorem, for every positive natural m one can construct a corresponding partially recursive function ω for which equations (1), (2), and (3) are identically satisfied. Each such function will be a principal universal function for partially recursive functions of one argument.

Let ω_0 be that function which is obtained from the recursion theorem for $m = 1$, i.e., that one of the functions ω satisfying equations (1), (2), and (3) for $m = 1$ which has the smallest domain of definition. According to what has been said, ω_0 will be a principal universal function for one-place partially recursive functions. Denote by (e) the following system of equalities of the formalism of recursive functions ((^{1}, § 54):

$$\begin{aligned} h(0,0,0,d) &= d, \\ h(a',0,c',d) &= h(a,c,0,d), \\ h(a',b',c,d) &= h(a,b,c',d), \\ f(a,b) &= h(a,c,0,h(d,c,b,d)), \\ f(a,b) &= h(a,0,c',c), \\ f(a,b) &= h(a,d',c',f(f(c,b),f(d,b))). \end{aligned} \tag{e}$$

Theorem 3. The system (e) recursively defines the function ω_0.

Corollary. If φ is an arbitrary partially recursive function, then a system of 7 equalities of the formalism of recursive functions can be constructed which recursively defines φ.

One can construct a relatively simple normal algorithm which, in a certain sense, computes the function ω_0. We have constructed a normal algorithm $\mathfrak{A}$ in the alphabet $|f0abc$, possessing the following properties (for the terminology see (2)):

1) For any natural v and z the equality holds

$$\mathfrak{A}\left[\, |^{v+1}0|^{z+1} \,\right] \simeq |^{\omega_0(v,z)+1}.$$

2) The scheme of the algorithm $\mathfrak{A}$ consists of 18 substitution formulas.
3) All formulas in the scheme of the algorithm $\mathfrak{A}$ are simple.
4) The length of the representation of the algorithm $\mathfrak{A}$ is equal to the number 128.

From the existence of the normal algorithm $\mathfrak{A}$ with properties 1), 2), and 3) it follows that, for any one-place partially recursive function φ, one can construct a normal algorithm $\mathfrak{B}$ over the alphabet $|$, possessing the following properties:

A. For any natural z the equality is true

$$\mathfrak{B}\left[\, |^{z} \,\right] \simeq |^{\varphi(z)}.$$

B. The scheme of the algorithm $\mathfrak{B}$ consists of 20 substitution formulas.

By applying Theorem 2 for $m = 2$, one can obtain certain results concerning relative partial recursiveness. We formulate here one of them:

Theorem 4. If φ and ψ are partial arithmetical functions and φ is partially recursive relative to ψ, then one can construct a system of 8 equalities of the formalism of recursive functions, recursively defining φ through ψ.

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Note: Figure translations are in progress. See original paper for figures.

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