A PROBLEM WITH TWO FREE BOUNDARIES FOR AN ELLIPTIC EQUATION AND OPTIMAL STOPPING OF A MARKOV PROCESS

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Abstract

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A PROBLEM WITH TWO FREE BOUNDARIES FOR AN ELLIPTIC EQUATION AND OPTIMAL STOPPING OF A MARKOV PROCESS

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1. Let E_n be n-dimensional Euclidean space, U a bounded domain in E_n with twice continuously differentiable boundary,

$$Lu(x) = a_{ij}(x)u_{ij}(x) + b_i(x)u_i(x) - c(x)u(x)$$

an elliptic differential operator acting on functions $u \in W_p^2(U)$ (see ([1])). Here u_{ij} is the mixed derivative with respect to x_i, x_j; u_i is the derivative with respect to x_i; summation over the indices i, j is assumed. Suppose that a_{ij}, b_i, c are bounded, $c \geq 0$, the matrix a_{ij} is uniformly nondegenerate, and, for $n \leq 2$, $p = 2$, while for $n > 2$ the $a_{ij}(x)$ are continuous in E_n, $p > n/2$. Let two functions $\psi_1, \psi_2 \in W_p^2(U)$ also be given such that $\psi_1 \geq \psi_2$ in U, $\psi_1 = \psi_2$ on ∂U.

Theorem 1. There exists, and moreover is unique, a function $u \in W_p^2(U)$ possessing the following two properties: a) $\psi_1(x) \geq u(x) \geq \psi_2(x)$ for $x \in U \cup \partial U$; b) $(-1)^{iLu} \geq 0$ (a.e.) on

$$\Gamma_i = \{x : \ (-1)^i[u(x) - \psi_i(x)] > 0\}$$

for $i = 1, 2$. In addition, if $p > n$, then $\operatorname{grad}(u - \psi_i) = 0$ at all points of the set $U \setminus \Gamma_i, \ i = 1, 2$.

The problem consisting in finding a function satisfying conditions a), b) is called a problem with two free boundaries, because when $\psi_1 > \psi_2$ in U it is equivalent to the problem of finding a function $u \in W_p^2(U)$ and two closed sets G_1, G_2 such that a) is satisfied, $u = \psi_1$ and $Lu \geq 0$ (a.e.) on G_1, $u = \psi_2$ and $Lu \leq 0$ (a.e.) on G_2, and

$$Lu = 0$$

(a.e.) on $U \setminus (G_1 \cup G_2)$.

This problem is a generalization of the problem with one free boundary and is directly related to problems with a variational inequality ([2]). Namely, if $L\psi_1 < 0$, then from Theorem 1 it is not difficult to derive that u is the unique function in $W_p^2(U)$ such that

$$u \geq \psi_2, \qquad (u - \psi_2)|_{\partial U} = 0,$$

$$Lu \leq 0$$

(a.e.) on $\{x : \; u(x) = \psi_2(x)\}$, and

$$Lu = 0$$

(a.e.) on $\{x : \; u(x) > \psi_2(x)\}$.

All these results can be obtained with the aid of certain facts from the theory of Markov processes. They are presented below.

2. Let X be a standard process in a locally compact space E (see ([3])), B the space of Borel bounded functions f with norm

$$|f| = \sup\{|f| : \; x \in E\},$$

R_λ the resolvent of the process X, m some probabilistic regular measure on E, $p > 1$. Suppose that there exists a constant N such that for all $x \in E$, $f \in B$ the function $R_0 f(x)$ is continuous and

$$R_0|f|(x) \leq N\|f\|_p,$$

where $\|f\|_p$ is the norm in the space L_p of functions integrable to the p-th power with respect to the measure m. Let two Borel functions $\psi_1(x), \psi_2(x)$ also be given, with $\psi_1 \geq \psi_2$. We want to study the function

$$v(x) = \inf_\tau \sup_\sigma M_x\{\psi_1(x_\tau)\chi_{\tau<\sigma} + \psi_2(x_\sigma)\chi_{\sigma \leq \tau}\},$$

where the lower and upper bounds are taken over the set of all Markov times τ and σ, respectively.

The problem of studying $v(x)$ is a generalization of the problems considered in ([4–6]). Our method differs from the methods known from ([4–6]); it consists in representing the function $v(x)$ in the form $R_0 f(x)$, where $f \in L_p$. In proving this representation we use the method of "continuous" stopping. The main technical tool is the following lemma.

For $c \in B$ put

$$R_c f(x) = M_x \int_0^\infty \exp\left[-\int_0^t c(x_s)\,ds\right] f(x_t)\,dt,$$

and, for $n, m > 0$, let

$$u_{nm} = \inf\sup R_{c_1+c_2}(c_1\psi_1 + c_2\psi_2),$$

where the inf(sup) is taken over the set of all functions $c_1(c_2)$ such that $0 \le c_1 \le n$ $(0 \le c_2 \le m)$,

$$\delta_{nm} = -(u_{nm} - \psi_1)_+, \qquad \rho_{nm} = (u_{nm} - \psi_2)_-{}^*.$$

Lemma. Suppose that, for some $f^i \in L_p$, $\psi_i = R_0 f^i$ $(i = 1, 2)$. Then:

a) $u_{nm} = R_c(n\delta_{nm} + m\rho_{nm})$;

b) if $n > m$, then

$$|\delta_{nm}| \le R_n f_-^1, \qquad \rho_{nm} \le R_m n R_n f_-^1 + R_m f_+^2;$$

c)

$$u_{nm}(x) = M_x\left[\int_0^{\tau\wedge\sigma} (n\delta_{nm} + m\rho_{nm})(x_t)\,dt + u_{nm}(x_{\tau\wedge\sigma})\right]$$

$$= \sup_\sigma M_x\left[\int_0^{\tau\wedge\sigma} n\delta_{nm}(x_t)\,dt - \rho_{nm}(x_\sigma)\chi_{\sigma\le\tau} + \psi_2(x_\sigma)\chi_{\sigma\le\tau} + u_{nm}(x_\tau)\chi_{\tau<\sigma}\right]$$

$$= \inf_\tau\sup_\sigma M_x\left[\psi_1(x_\tau)\chi_{\tau<\sigma} + \psi_2(x_\sigma)\chi_{\sigma\le\tau} - \delta_{nm}(x_\tau)\chi_{\tau<\sigma} - \rho_{nm}(x_\sigma)\chi_{\sigma\le\tau}\right].$$

Proof. Consider the operator

$$F_{nm}v = R_{n+m}[(n+m)v - n(v-\psi_1)_+ + m(v-\psi_2)_-]$$

in the space B. From the inequality

$$\left|[(m+n)t_1 - n(t_1-a)_+ + m(t_1-b)_-] - [(m+n)t_2 - n(t_2-a)_+ + m(t_2-b)_-]\right| \le (n+m)|t_1-t_2|$$

it follows that

$$|F_{nm}v_1 - F_{nm}v_2| \le (n+m)R_{n+m}|v_1 - v_2|.$$

Further, note that from the estimate $|R_0 f| \leq N\|f\|$ it follows that

$$(n+m)\|R_{n+m} f\| \leq \varepsilon_{n+m}\|f\|,$$

where $\varepsilon_{n+m} < 1$ and does not depend on f.

Thus the operator F_{nm} is a contraction in B. By Banach' s theorem there exists $v_{nm} \in B$ such that

$$v_{nm} = F_{nm} v_{nm}.$$

Next, for $0 \leq d \leq k$ the equality

$$R_d f = \sum_{i=0}^{\infty} [R_k(k-d)]^i R_k f$$

holds. Hence, for

$$d = c_1 + c_2, \qquad k = n+m, \qquad f = dv_{nm} - n(v_{nm} - \psi_1)_+ + m(v_{nm} - \psi_2)_-,$$

in view of the equality

$$R_{n+m} f = c_{nm} + R_{n+m}(d-k) v_{nm},$$

it easily follows that

$$R_{c_1+c_2} f = v_{nm}$$

for $0 \leq c_1 \leq n,\ 0 \leq c_2 \leq m$.

From this equality and the inequality

$$c_2(v_{nm} - \psi_2) + m(v_{nm} - \psi_2)_- \geq 0$$

we obtain

$$v_{nm} = \sup R_{c_1+c_2}[c_1 v_{nm} - n(v_{nm} - \psi_1)_+ + c_2\psi_2],$$

and, as a consequence of the inequality

$$c_1(v_{nm} - \psi_1) - n(v_{nm} - \psi_1)_+ \leq 0,$$

this gives

$$v_{nm} = \inf\sup R_{c_1+c_2}(c_1\psi_1 + c_2\psi_2),$$

i.e. $v_{nm} = u_{nm}$. Assertion a) now follows from the equality

$$R_{c_1+c_2} f = v_{nm}$$

for $c_1 = c_2 = 0$.

Let us again turn to the equality

$$u_{nm} = R_{c_1+c_2}[(c_1+c_2)u_{nm} + n\delta_{nm} + m\rho_{nm}].$$

For $n > m,\ a \geq b$ we have

$$nt - n(t-a)_+ + m(t-b)_- \leq na,$$

therefore, for $c_1 = n,\ c_2 = 0,\ n > m$, we obtain

$$u_{nm} \leq nR_n\psi_1,$$

and since

$$nR_n\psi_1 = nR_{nR}0f^1 = R_0f^1 - R_nf^1 = \psi_1 - R_nf^1,$$

it follows that

$$u_{nm} \leq \psi_1 - R_nf^1 \leq \psi_1 + R_nf^1_-$$

and

$$|\delta_{nm}| \leq R_nf^1_-.$$

If, however, we take $c_1 = 0,\ c_2 = m$, then

$$u_{nm} \geq -R_mnR_nf^1_- + R_m[mu_{nm} + m\rho_{nm}] \geq -R_mnR_nf^1_- + mR_m\psi_2,$$

and

$$\rho_{nm} \leq R_mnR_nf^1_- + R_mf^2_+.$$

Assertion b) is proved.

The first equality in c) follows from a) and the strong Markov property of X. The second equality is valid because, on the one hand,

$$u_{nm} \geq \psi_2 - \rho_{nm}$$

and $\rho_{nm} \geq 0$, while, on the other hand, if

$$\sigma = \inf\{t : \rho_{nm}(x_t) > 0\},$$

then

$$\int_0^{\tau\wedge\sigma} \rho_{nm}(x_t)\, dt = 0$$

and

$$\psi_2(x_\sigma) - \rho_{nm}(x_\sigma) = u_{nm}(x_\sigma).$$

The third equality is obtained by analogous arguments from the second. The lemma is proved.

From this lemma the following theorems are derived rather simply. In their statements,

$$M_x(\tau,\sigma) = M_x[\psi_1(x_\tau)\chi_{\tau<\sigma} + \psi_2(x_\sigma)\chi_{\sigma\leq\tau}].$$

Theorem 2. Suppose that, for some $f^i \in B$, $\psi_i = R_2f^i$ $(i = 1,2)$; then there exists a function $f \in B$ such that $v = R_0f$, and hence v is continuous. Further,

$$(-1)^if \leq 0 \quad \text{on} \quad \Gamma_i(v) = \{x : (-1)^i[v(x) - \psi_i(x)] > 0\},$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\|v-u_{nm}\|=0,\qquad v(x)=\sup_{\sigma}M_x(\tau_1,\sigma)=\inf_{\tau}M_x(\tau,\tau_2),$$

$$^{*}\quad a_+=\frac{1}{2}(|a|+a),\qquad a_-=\frac{1}{2}(|a|-a).$$

where $\tau_i=\inf\{t:\ v(x_t)=\psi_i(x_t)\}$ $(i=1,2)$. Moreover, if $u,h\in B$ are functions related by the relation $u=R_0h$, $\psi_1\geq u\geq\psi_2$, and $(-1)^ih\leq 0$ on $\Gamma_i(u)$, for $i=1,2$, then $u=v$.

Theorem 3. Suppose there exists a constant N_1 such that $\lambda\|R_\lambda f\|_p\leq N_1\|f\|_p$ for all $f\in B$, $\lambda>0$. Then in the formulation of Theorem 2 one may replace B by L_p. Moreover, as τ_i one may also take

$$\inf\{t:\ v(x_t)=\psi_i(x_t),\quad x_t\in\operatorname{supp}[(-1)^if]^+\}.$$

Before formulating Theorem 4, take two closed sets G_1,G_2 and denote by $\mathfrak{M}_i$ the set of Markov times such that

$$P_x\{x_\tau\in G_i\}=P_x\{\tau<\zeta\}$$

for all x, and by $\mathfrak{N}_i$ the set of hitting times of closed subsets of G_i.

Theorem 4. Suppose the assumption of Theorem 3 is satisfied and there exist sequences of functions $f_n^i\in L_p$ such that

$$\operatorname{supp}[(-1)^if_n^i]^+\subset G_i,\qquad R_0f_n^1\geq R_0f_n^2,\qquad \lim_{n\to\infty}\|[\psi_i-R_0f_n^i]\chi_{G_i}\|=0.$$

Then

$$\inf_{\tau\in\mathfrak{M}_1}\times\sup_{\sigma\in\mathfrak{M}_2}M_x(\tau,\sigma)=\inf_{\tau\in\mathfrak{N}_1}\sup_{\sigma\in\mathfrak{M}_2}M_x(\tilde{\tau},\sigma)=\inf_{\tau\in\mathfrak{M}_1}\sup_{\sigma\in\mathfrak{N}_2^x}M_x(\tau,\sigma).$$

Moreover, all expressions in the last equality are continuous.

We make some remarks concerning the proofs of Theorems 1-4. Theorem 1 is derived from Theorem 3 with the aid of the known properties of R_0f for quasidiffusion processes; Theorem 4 is also easily derived from Theorem 3. The proofs of Theorems 2 and 3 follow one and the same plan. First, with the aid of Lemma b), it is proved that

$$\lim_{n\to\infty}\lim_{m\to\infty}\|\rho_{nm}+|\delta_{nm}|\|=0;$$

then, using Lemma c), that $\|u_{nm}-v\|\to 0$. After this, from the estimate $R_0|f|\leq N\|f\|_p$ and Lemmas a), b), one obtains the equality $v=R_0f$ and the sign properties of $(-1)^if$. The equality

$$v(x)=\sup_{\sigma}M_x(\tau_1,\sigma)$$

follows from the equality of u_{nm} to the third term in equality c) of the lemma and from the estimates $|\delta_{nm}| \leq R_n f^-$, $R_0|f| \leq N\|f\|_p$, $\lambda\|R_\lambda f\|_p \leq N_1\|f\|_p$. The last assertion of Theorem 2 is almost obvious.

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CITED LITERATURE

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

2. H. Lewy, Comm. Pure Appl. Math., **22**, 2, 153 (1969).

3. E. B. Dynkin, Markov Processes, M., 1963.

4. E. B. Dynkin, DAN, **150**, No. 2, 238 (1963).

5. B. I. Grigelionis, A. N. Shiryaev, Theory of Probability and Its Applications, **11**, 4, 612 (1966).

6. S. M. Gusein-Zade, Theory of Probability and Its Applications, **14**, 2, 732 (1969).

Note: Figure translations are in progress. See original paper for figures.