Mechanics

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Abstract

Full Text

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ON OSCILLATIONS OF SYSTEMS CLOSE TO ESSENTIALLY NONLINEAR ONES

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The paper considers oscillations of systems close to essentially nonlinear ones, in the special case when the equations for determining the principal amplitudes are identically satisfied for any choice of part of the constants. It turns out that one part of the arbitrary constants entering each approximation is determined from the next approximation, and the other part from the approximation one order higher.

1. Formulation of the problem. Oscillations of systems close to essentially nonlinear ones are described by equations of the form

$$\dot{x}_s = X_s(t, x_1, \dots, x_n) + \mu f_s^{(1)}(t, x_1, \dots, x_n) + \mu^2 f_s^{(2)}(t, x_1, \dots, x_n) + \cdots \quad (s = 1, 2, \dots, n), \tag{1}$$

where the right-hand sides are periodic with respect to t with period ω and analytic with respect to $x_1, \dots, x_n, \mu$, and X_s are essentially nonlinear functions of $x_1, x_2, \dots, x_n$; μ is a small parameter.

Suppose that the generating system

$$\dot{x}_s^0 = X_s(t, x_1^0, \dots, x_n^0) \tag{2}$$

admits a family of periodic solutions of period ω

$$x_s^0 = \varphi_s = \varphi_s(t, h_1, \dots, h_m), \tag{3}$$

depending on m arbitrary constants $h_1, h_2, \dots, h_m$ and lying in the domain of definition of the variables $t, \mu, x_1, x_2, \dots, x_n$.

Then, as shown in ([1]), system (1) admits a solution, periodic and analytic with respect to μ, which for $\mu = 0$ becomes the solution $\varphi_s(t, h_1^*, h_2^*, \dots, h_m^*)$ of the family (3). The constants $h_1^*, \dots, h_m^*$ must satisfy the equations

$$P_i(h_1^*, \dots, h_m^*) = \int_0^\omega \sum_{s=1}^n f_s^{(1)}[t, \varphi_1(t, h_1^*, \dots, h_m^*), \dots, \varphi_n(t, h_1^*, \dots, h_m^*)]\psi_{si}\, dt = 0, \tag{4}$$

where ψ_{si} are periodic solutions of the equations conjugate to the variational equations

$$\dot{\psi}_{si} = -\sum_{\alpha=1}^n p_{\alpha s}\psi_{\alpha i}.$$

We shall investigate the special case when equations (4) are satisfied identically for any choice of part of the constants $h_1^*, h_2^*, \dots, h_r^*$ $(0 \le r \le m)$, and we shall prove that, under certain conditions, there exists a unique system of series of the form

$$x_s = \varphi_s(t, h_1^*, \dots, h_m^*) + \mu x_s^{(1)} + \mu^2 x_s^{(2)} + \cdots \quad (s = 1, 2, \dots, n), \tag{5}$$

satisfying equations (1), where $x_s^{(p)}$ are certain periodic functions of t of period ω.

2. Main theorem. To obtain the equations for the unknown functions $x_s^{(p)}$, we equate the coefficients of like powers of μ in the original equations (1), replacing x_s by their expressions (5). As a result we obtain:

$$\dot{x}_s^{(1)} = \sum_{\alpha=1}^n p_{s\alpha}x_\alpha^{(1)} + f_s^{(1)}(t, \varphi_1, \dots, \varphi_n), \tag{6}$$

$$\dot{x}_s^{(2)} = \sum_{\alpha=1}^n p_{s\alpha}x_\alpha^{(2)} + \frac{1}{2}\sum_{\alpha,\beta=1}^n \frac{\partial^2 X_s(t, \varphi_1, \dots, \varphi_n)}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(1)} x_\beta^{(1)} + \sum_{\alpha=1}^n \frac{\partial f_s^{(1)}(t, \varphi_1, \dots, \varphi_n)}{\partial\varphi_\alpha} x_\alpha^{(1)} + f_s^{(2)}(t, \varphi_1, \dots, \varphi_n),$$

$$\begin{aligned}\dot{x}_s^{(3)} = &\sum_{\alpha=1}^n p_{s\alpha}x_\alpha^{(3)} + \sum_{\alpha,\beta=1}^n \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(1)} x_\beta^{(2)} + \frac{1}{6}\sum_{\alpha,\beta,\gamma=1}^n \frac{\partial^3 X_s}{\partial\varphi_\alpha \partial\varphi_\beta \partial\varphi_\gamma} x_\alpha^{(1)} x_\beta^{(1)} x_\gamma^{(1)} \\ &+ \sum_{\alpha=1}^n \frac{\partial f_s^{(1)}}{\partial\varphi_\alpha} x_\alpha^{(2)} + \frac{1}{2}\sum_{\alpha,\beta=1}^n \frac{\partial^2 f_s^{(1)}}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(1)} x_\beta^{(1)} + \sum_{\alpha=1}^n \frac{\partial f_s^{(2)}}{\partial\varphi_\alpha} x_\alpha^{(1)} + f_s^{(3)}(t, \varphi_1, \dots, \varphi_n),\end{aligned} \tag{7}$$

$$\dot{x}_s^{(k+2)} = \sum_{\alpha=1}^{n} p_{s\alpha} x_\alpha^{(k+2)} + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(k+1)} x_\beta^{(1)} + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(k)} x_\beta^{(2)}$$
$$+ \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f_s^{(1)}}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(1)} x_\beta^{(k)} + \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial^3 X_s}{\partial\varphi_\alpha \partial\varphi_\beta \partial\varphi_\gamma} x_\alpha^{(k)} x_\beta^{(1)} x_\gamma^{(1)} \quad (8)$$
$$+ \sum_{\alpha=1}^{n} \frac{\partial f_s^{(1)}}{\partial\varphi_\alpha} x_\alpha^{(k+1)} + \sum_{\alpha=1}^{n} \frac{\partial f_s^{(2)}}{\partial\varphi_\alpha} x_\alpha^{(k)} + F_s^{(k+2)}$$

$$(s = 1, 2, \dots, n;\ k > 1),$$

where $p_{s\alpha} = \partial X_s(t, \varphi_1, \dots, \varphi_n)/\partial\varphi_\alpha$, and $F_s^{(k+2)}$ are completely determined periodic functions of time, independent of $x_\alpha^{(u)}$ $(u \geq k)$.

A periodic solution of the equations with respect to $x_s^{(1)}$ (6), when conditions (4) are satisfied, may be represented in the form

$$x_s^{(1)} = x_s^{(1)*} + \sum_{j=1}^{m} M_j^{(1)} \varphi_{sj},$$

where $M_j^{(1)}$ are arbitrary constants, $x_s^{(1)*}$ is some particular periodic solution of equations (6), and $\varphi_{s1}, \dots, \varphi_{sm}$ are periodic solutions of the variational equations $\dot{\varphi}_{sj} = \sum_{\alpha=1}^{n} p_{s\alpha}\varphi_{\alpha j}$. These solutions are connected with φ_j by relations of the form $\varphi_{sj} = \partial\varphi_s(t, h_1^*, \dots, h_m^*)/\partial h_j^*$. Thus, the functions $x_s^{(1)}$ contain $m + r$ arbitrary constants $h_1^*, \dots, h_r^*, M_1^{(1)}, \dots, M_m^{(1)}$.

In order that equations (7) admit a periodic solution with respect to $x_s^{(2)}$, it is necessary and sufficient that the conditions

$$Q_i(h_1^*, \dots, h_r^*, M_1^{(1)}, \dots, M_m^{(1)}) = \int_0^\omega \sum_{s=1}^{n} \left\{ \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\alpha^{(1)} x_\beta^{(1)} + \sum_{\alpha=1}^{n} \frac{\partial f_s^{(1)}}{\partial\varphi_\alpha} x_\alpha^{(1)} + f_s^{(2)}(t, \varphi_1, \dots, \varphi_n) \right\} \psi_{si}\, dt = 0 \quad (9)$$

$$(i = 1, 2, \dots, m)$$

Theorem. If the functional determinant of the left-hand sides of equations (9) with respect to h_l^* and $M_q^{(1)}$ $(l = 1, 2, \dots, r;\ q = r + 1, \dots, m)$ is nonzero, then there exists a unique system of series of the form (5) satisfying equations (1).

As in the author' s preceding paper [2], the course of the proof of this theorem is as follows. Suppose that all the functions $x_s^{(h)}$ up to order $(k+1)$ have already been computed and have turned out to be periodic:

$$x_s^{(h)} = x_s^{(h)*} + \sum_{j=1}^{m} M_j^{(h)} \varphi_{sj} \qquad (h = 1, 2, \dots, k+1), \tag{10}$$

where $x_s^{(h)*}$ is some particular periodic solution of the equations for $x_s^{(h)}$, and $M_j^{(h)}$ are constants. Assume that the functions $x_s^{(1)}, \dots, x_s^{(k-1)}$ have been determined completely together with the constants $M_1^{(u)}, \dots, M_m^{(u)}$ $(u = 1, 2, \dots, k-1)$ entering into them from the periodicity conditions for the functions $x_s^{(2)}, \dots, x_s^{(k+1)}$. The constants $M_{r+1}^{(k)}, \dots, M_m^{(k)}$ have also been determined from the periodicity conditions for the functions $x_s^{(k+1)}$. The quantities $M_l^{(k)}$ and $M_q^{(k+1)}$ $(l = 1, 2, \dots, r;\ q = r+1, \dots, m)$ still remain to be determined.

To find these constants, we write down the periodicity conditions for the functions $x_s^{(k+2)}$, which, after simple intermediate transformations, take the form

$$\begin{aligned}\Omega_i = &\sum_{j=1}^{m} \left[\int_0^{\omega} \sum_{s=1}^{n} \left\{ \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\beta^{(1)} + \sum_{\alpha=1}^{n} \frac{\partial f_s^{(1)}}{\partial\varphi_\alpha} \right\} \varphi_{\alpha j} \psi_{si}\, dt \right] M_j^{(k+1)} \\ &+ \sum_{\rho=1}^{m} \left[\int_0^{\omega} \sum_{s,\alpha=1}^{n} \left\{ \sum_{\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\beta^{(2)*} \varphi_{\alpha\rho} + \frac{1}{2} \sum_{\beta,\gamma=1}^{n} \frac{\partial^3 X_s}{\partial\varphi_\alpha \partial\varphi_\beta \partial\varphi_\gamma} x_\beta^{(1)} x_\gamma^{(1)} \varphi_{\alpha\rho} \right.\right. \\ &\left.\left. + \sum_{\beta=1}^{n} \frac{\partial^2 f_s^{(1)}}{\partial\varphi_\alpha \partial\varphi_\beta} x_\beta^{(1)} \varphi_{\alpha\rho} + \frac{\partial f_s^{(2)}}{\partial\varphi_\alpha} \varphi_{\alpha\rho} + \sum_{\beta=1}^{n} \frac{\partial^2 X_s}{\partial\varphi_\alpha \partial\varphi_\beta} x_\beta^{(1)} \frac{\partial x_\alpha^{(1)}}{\partial h_\rho^*} + \frac{\partial f_s^{(1)}}{\partial\varphi_\alpha} \frac{\partial x_\alpha^{(1)}}{\partial h_\rho^*} \right\} \psi_{si}\, dt \right] \\ &\times M_\rho^{(k)} + R_i = 0 \qquad (i = 1, 2, \dots, m),\end{aligned} \tag{11}$$

where R_i are completely determined constants not containing $M_\rho^{(k)}$, $M_j^{(k+1)}$.

It is not difficult to show that equations (11) may be written in the form:

$$\Omega_i = \sum_{q=r+1}^{m} \frac{\partial Q_i}{\partial M_q^{(1)}} M_q^{(k+1)} + \sum_{l=1}^{r} \frac{\partial Q_i}{\partial h_l^*} M_l^{(k)} + R_i^* = 0 \qquad (i = 1, 2, \dots, m).$$

Here

$$R_i^* = R_i + \sum_{q=r+1}^{m} \frac{\partial Q_i}{\partial h_q^*} M_q^{(k)}$$

are known constants.

Thus we have obtained a system of m linear algebraic equations for determining the m constants $M_1^{(k)}, \dots, M_r^{(k)}, M_{r+1}^{(k+1)}, \dots, M_m^{(k+1)}$, whose determinant

$$\frac{\partial(Q_1, Q_2, \dots, Q_m)}{\partial(h_1^*, \dots, h_r^*, M_{r+1}^{(1)}, \dots, M_m^{(1)})}$$

is nonzero by assumption. Therefore the constants $M_l^{(k)}$, $M_q^{(k+1)}$ and, consequently, the functions $x_s^{(p)}$ in formulas (5) are determined uniquely, as was required to prove.

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REFERENCES

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2. Nguyen Van Dao, *Differential Equations*, Minsk, No. 7 (1968).

Note: Figure translations are in progress. See original paper for figures.