ON THE CLASSIFICATION OF TRISYMMETRIC SPACES

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Abstract

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ON THE CLASSIFICATION OF TRISYMMETRIC SPACES

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In the present note all nontrivial trisymmetric spaces ([1]) with simple compact Lie groups of motions will be indicated.

As is known ([1]), a homogeneous Riemannian space $V = Q/H$ is called trisymmetric if through each of its points x there pass three pairwise orthogonal mirrors $W_x^{(1)}, W_x^{(2)}, W_x^{(3)}$, and locally (in some neighborhood Ω_x of the point x)

$$W_x^{(1)} \cap W_x^{(2)} \cap W_x^{(3)} = \{x\}, \quad \dim V = \sum_{\alpha=1}^{3} \dim W_x^{(\alpha)}.$$

If, moreover, there exists $p \in \mathrm{Aut}(Q)$, $p^3 = I$, such that

$$pW_x^{(1)} = W_x^{(2)}, \quad pW_x^{(2)} = W_x^{(3)}, \quad pW_x^{(3)} = W_x^{(1)},$$

then we shall say that $V = Q/H$ is supertrisymmetric. A mirror ([2]) at a point $x \in V = Q/H$ is the maximal set of points $y \in V$ fixed with respect to an isometric subsymmetry ([2]) S_x $((S_x)^2 = I, S_x(x) = x)$. A mirror is a totally geodesic surface in the Riemannian $V = Q/H$ ([1,2]). If Γ is the Lie algebra of the group Q, then we introduce the notation

$$\Gamma \overset{\mathrm{def}}{=} \ln(Q);$$

if $L \subseteq \Gamma$ is a subalgebra, and $H \subseteq Q$ is the subgroup corresponding to it, then we shall use the notation

$$L \overset{\mathrm{def}}{=} \ln_Q(H).$$

A notation of the form Q/H means that this is the homogeneous space generated by the group Q and the subgroup H, but such that H may contain a nontrivial normal divisor of the group Q. The equality

$$Q'/H' = Q''/H''$$

means an isomorphism $Q' \cong Q''$ inducing an isomorphism $H' \cong H''$. A notation of the form Q/H means that this is the homogeneous space generated by the group Q and the subgroup H, where H contains only the trivial normal divisor of the group Q. Finally, the notation

$$Q'/H' \underset{\varphi}{=} Q''/H''$$

means the existence of a homomorphism φ such that $Q' \xrightarrow{\varphi} Q''$, $H' \xrightarrow{\varphi} H''$, and $\ker\varphi$ is the maximal normal divisor of the group Q' belonging to H'.

Definition 1. We shall say that a Lie group Q is an **invoproduct** of the groups Q_1, Q_2, Q_3, and write

$$Q = Q_1 \boxtimes Q_2 \boxtimes Q_3$$

(or, more briefly,

$$Q = \boxtimes_{\alpha=1}^{3} Q_\alpha$$

), if Q_α is the maximal subgroup of fixed elements of an involutive automorphism S_α ($(S_\alpha)^2 = I$) of the group Q, with

$$S_\lambda S_\mu = S_\nu \quad (\lambda \neq \mu,\ \mu \neq \nu,\ \nu \neq \lambda).$$

If, moreover, there exists $p \in \mathrm{Aut}(Q)$ such that

$$pQ_1 = Q_2, \quad pQ_2 = Q_3, \quad pQ_3 = Q_1,$$

then we shall call the invoproduct $\boxtimes_{\alpha=1}^{3} Q_\alpha$ a **superinvoproduct**.

Definition 2. We shall say that a homogeneous space Q/H is an **invoproduct of homogeneous spaces**

$$Q_\alpha^*/H_\alpha \quad (\alpha = 1, 2, 3)$$

and write

$$Q/H = Q_1^*/H_1 \boxtimes Q_2^*/H_2 \boxtimes Q_3^*/H_3$$

(or, briefly, $\cdots$).

In short, $Q/H = \prod_{\alpha=1}^{3} Q_\alpha/^* H_\alpha$, if $Q = \prod_{\alpha=1}^{3} Q_\alpha$, $H = \prod_{\alpha=1}^{3} H_\alpha$. In this case $Q_\alpha/^* H_\alpha$ will be called the **mirrors** in Q/H. We note that $Q_\alpha/^* H_\alpha$ are mirrors in Q/H in the sense of ([2]).

We now reformulate the definition of a trisymmetric homogeneous space, using the notion of an involutive product.

Definition 3. A homogeneous space $Q/H = \prod_{\alpha=1}^{3} Q_\alpha/^* H_\alpha$ is a **trisymmetric space** if (locally) $\bigcap_{\alpha=1}^{3} Q_\alpha = Q_0 \subseteq H$, moreover trivial if $Q_0 = H$, semitrivial if $Q_0 = H_\alpha$ (for some α), and nontrivial in the remaining cases. If, in addition, $\prod_{\alpha=1}^{3} Q_\alpha$ and $\prod_{\alpha=1}^{3} H_\alpha$ are superinvolutive products with respect to $p \in \mathrm{Aut}(Q)$, then we shall say that Q/H is **supertrisymmetric**. We note that the mirrors in a trisymmetric space are symmetric spaces.

We formulate two auxiliary theorems.

Theorem 1. Let Q be simple, and let $Q/H = \prod_{\alpha=1}^{3} Q_\alpha /^* H_\alpha$ be a nontrivial trisymmetric space with compact group H; then

$$Q_\alpha /^* H_\alpha = (\widetilde{Q}_\alpha \times \widetilde{H}_\alpha)/(\widetilde{Q}_\alpha \cap \widetilde{H}_\alpha) \times \widetilde{H}_\alpha = \widetilde{Q}_\alpha/(\widetilde{Q}_\alpha \cap H_\alpha),$$

where $\widetilde{Q}_\alpha \cap H_\alpha \subseteq \bigcap_{\alpha=1}^{3} Q_\alpha = Q_0$, and $\widetilde{H}_\alpha$ is a normal divisor in Q_α.

Theorem 2. Let $Q/H = \prod_{\alpha=1}^{3} Q_\alpha /^* H_\alpha$ be a trisymmetric space with compact group H, and suppose the symmetric space Q_α^*/Q_0 $(Q_0 = \bigcap_{\alpha=1}^{3} Q_\alpha)$ is irreducible for some α; then Q/H is trivial or semitrivial.

The problem of classifying trisymmetric spaces

$$Q/H = \prod_{\alpha=1}^{3} Q_\alpha /^* H_\alpha$$

is naturally solved by passing from groups to the corresponding Lie algebras. Then the involutive products $Q = \prod_{\alpha=1}^{3} Q_\alpha$ and $H = \prod_{\alpha=1}^{3} H_\alpha$ generate involutive sums ([3,5]) of Lie algebras

$$\ln_Q(Q) = \ln_Q(Q_1) + \ln_Q(Q_2) + \ln_Q(Q_3), \qquad \ln_Q(H) = \ln_Q(H_1) + \ln_Q(H_2) + \ln_Q(H_3),$$

where

$$L_0 = \ln_Q\left(\bigcap_{\alpha=1}^{3} Q_\alpha\right) \subseteq \ln_Q(H).$$

Taking Theorems 1 and 2 into account, we also obtain, for a nontrivial trisymmetric space Q/H, that

$$\ln_Q(Q_\alpha)/^* L_0$$

$(\alpha = 1, 2, 3)$ are reducible involutive pairs ([3,4]). Thus, first of all, one should seek involutive decompositions of simple compact Lie algebras satisfying the condition stated above. Next one should seek nontrivial ideals M_α in

$$L_\alpha = \ln_Q(Q_\alpha) \qquad (\alpha = 1, 2, 3)$$

so that

$$K = M_1 + M_2 + M_3 + L_0$$

is a subalgebra in $\Gamma = \ln Q$. Such a subalgebra also determines $K = \ln_Q(H)$. Finally, from the pair of algebras Γ/K we uniquely (locally) reconstruct the trisymmetric space Q/H, and from the involutive pairs ([3,4])

$$L_\alpha /^* L_0$$

we reconstruct its mirrors $Q_\alpha /^* H_\alpha$. In the cases of the classical simple algebras $so(n)$, $su(n)$, $sp(n)$, the problem is solved using known matrix models. The case of the exceptional algebras g_2, f_4, e_6, e_7, e_8, owing to the absence of good matrix models, is considerably more difficult. With the aid of the apparatus of involutive sums ([3–6]) and of a certain procedure for reconstructing involutive sums, all the data needed for the classification of trisymmetri-

···spaces the involutive decompositions of exceptional algebras can be found. Thus, if $\Gamma = L_1 + L_2 + L_3,\ L_0 = L_\alpha \cap L_\beta$ $(\alpha \neq \beta)$, then for the involutive pairs Γ/L_α and the reducible pairs L_α/L_0, respectively, we have:

1) $g_2/so(4),\ so(4)/so(2) \oplus so(2);$

2) $f_4/su(2) \oplus sp(3),\ su(2) \oplus sp(3)/u(1) \oplus u(3);$

3) $e_6/su(2) \oplus su(6),\ su(2) \oplus su(6)/u(1) \oplus s(u(3) \oplus u(3));$

4) $e_7/su(2) \oplus so(12),\ su(2) \oplus so(12)/u(1) \oplus u(6);$

5) $e_7/u(1) \oplus e_6,\ u(1) \oplus e_6/f_4;$

6) $e_8/su(2) \oplus e_7,\ su(2) \oplus e_7/u(1) \oplus u(1) \oplus e_6.$

We note that in 1), 2), 3), 4), 6) Γ/L_α are principal unitary involutive pairs ([4]), while in 5) Γ/L_α is a central involutive pair ([4]); moreover, all the decompositions obtained are superinvolutive ([4]). Using the decompositions obtained for exceptional algebras and matrix models of classical algebras and their involutive automorphisms, we arrive at a complete classification of nontrivial trisymmetric and nonsymmetric spaces

$$Q/H = \mathop{\boxtimes}_{\alpha=1}^{3} Q^*_\alpha/H_\alpha$$

with simple compact groups Q of motions and maximal rotation groups

$Q/H = \boxtimes_{\alpha=1}^{3} Q^*_\alpha/H_\alpha$	Q^*_α/H_α
$G_2/SU(3)$	$SO(4)^*/SU(2) \times SO(2) = SU(2)/SO(2)$
$F_4/SU(3) \times SU(3)$	$SU(2) \times Sp(3)^*/SU(2) \times U(3) = Sp(3)/U(3)$
$E_6/SU(3) \times SU(3) \times SU(3)$	$SU(2) \times SU(6)^*/SU(2) \times S(U(3) \times U(3)) = SU(6)/S(U(3) \times U(3))$
$E_7/SU(6) \times SU(3)$	$SU(2) \times SO(12)^*/SU(2) \times U(6) = SO(12)/U(6)$
$E_7/F_4 \times SO(3)$	$E_6 \times U(1)^*/F_4 \times U(1) = E_6/F_4$
$E_8/SU(3) \times E_6$	$SU(2) \times E_7^*/SU(2) \times E_6 \times U(1) = E_7/E_6 \times U(1)$
$SO(4m)/Sp(m) \times SO(3)$	$U(2m)^*/Sp(m) \times U(1) = SU(2m)/Sp(m)$

$Q/H = \boxtimes_{\alpha=1}^{3} Q_\alpha^*/H_\alpha$	Q_α^*/H_α
$SU(2m)/SU(m) \times SO(3)$	$S(U(m) \times U(m))^*/SU(m) \times U(1) =$ $SU(m) \times SU(m)/SU(m)$
$Sp(m)/SO(m) \times SO(3)$	$U(m)^*/SO(m) \times U(1) =$ $SU(m)/SO(m)$

with natural embeddings.

All the spaces obtained are supertrisymmetric with principal unitary or central mirrors (i.e. with mirrors generated by principal unitary or central automorphisms ([4,6]) of the group Q); moreover, the conjugating automorphism p belongs to $SU(3)$—the normalizer of the rotation group $\widetilde{H}$ in the case of a principal mirror—and p belongs to $SO(3)$—the normalizer of the rotation group $\widetilde{H}$ in the case of a central mirror.

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Note: Figure translations are in progress. See original paper for figures.