A NEUTRINO MODEL IN GENERAL RELATIVITY WITH MOTION GROUPS \(G_3\) ON \(V_2\)

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Abstract

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A NEUTRINO MODEL IN GENERAL RELATIVITY WITH MOTION GROUPS G_3 ON V_2

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1. Let us consider a neutrino model in general relativity which is described by the Einstein–Dirac system of equations [1]

$$\tilde{\gamma}^{ia}_{\ \ b}\nabla_i\psi^b = 0, \qquad R_{ij} = \varkappa T_{ij}, \tag{1}$$

where the energy-momentum tensor is

$$T_{ij} = \mathrm{Re}\left(\psi^+_a\gamma^a_{(ib}\Delta_{j)}\psi^b\right)^*. \tag{2}$$

R_{ij} is the Ricci tensor of a four-dimensional Riemannian manifold of index 1 with metric tensor g_{ij}. The spinor ψ^a $(a = 1, 2, 3, 4)$ is defined in a nonholonomic coordinate system consisting of orthonormal frames b^i_α, b^α_i, $b^\alpha_j g_{ij} = \tilde{g}_{\alpha\beta}$, and is transformed according to a representation of the Lorentz group determined by the invariance of the spin-tensor $\tilde{\gamma}^{\alpha a}_{\ \ b}$ **:

$$\tilde{\gamma}^1 = \left\|\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right\|, \qquad \tilde{\gamma}^2 = \left\|\begin{array}{cc|cc} 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \\ \hline -i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{array}\right\|,$$

$$\tilde{\gamma}^3 = \left\|\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right\|, \qquad \tilde{\gamma}^4 = \left\|\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right\|. \tag{3}$$

Lowering of indices is carried out with the aid of the invariant spin-tensor e_{ab}; contraction is always over the second index. The conjugate spinor ψ^+_a is introduced with the aid of the spin-tensor Π_{ba}: $\psi^+_a = \psi^{\dot{b}}\Pi_{ba}$, where the dot denotes complex conjugation,

$$e = \left\| \begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right\|, \qquad \Pi = \left\| \begin{array}{cc|cc} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ \hline 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{array} \right\|; \tag{4}$$

$$\nabla_i \psi^a = \partial_i \psi^a + \Gamma^a_{ib} \psi^b,$$

where parallel transport of the spinor is induced by

* The model under consideration satisfies the general equation in variations [2,3]

$$\delta \int_{V_4} \Lambda \, d\tau + \delta W = 0 \quad \text{for} \quad \Lambda = -R/2\varkappa + \mathrm{Re}\,(\psi^+_a \gamma^{ia}{}_b \nabla_i \psi^b)\,.$$

** The letters a, b, c denote spinor indices; the tilde indicates that the quantity is referred to a nonholonomic orthogonal coordinate system.

by parallel transport of the frame ([3])

$$\Gamma^a_{ic} = \frac{1}{4} \gamma^a{}_b \gamma^b_{jc} \left(b^j_\alpha \partial_i a^\alpha_k - \Gamma^j_{ik} \right), \tag{5}$$

$$b^i_\alpha a^\alpha_j = \delta^i_j, \quad \partial_i = \partial / \partial x^i,$$

Γ^j_{ik} are the coefficients of the Riemannian connection. $c = \hbar = 1$. For $\psi^1 = \psi^2 = 0$ equations (1) and (2) reduce to the Einstein–Weyl equations ([4]) for a 2-component neutrino, and for $\psi^3 = \psi^4 = 0$, for an antineutrino.

It is known ([5]) that a spinor can be represented by a system of real tensors

$$\Omega = \psi^\dagger_a \psi^a, \qquad J^k = -i \psi^\dagger_a \gamma^{ka}{}_b \psi^b, \qquad M^{kj} = -i \psi^\dagger_a \gamma^{[ka}{}_b \gamma^{j]b}{}_c \psi^c,$$

$$S^{ijk} = \psi^\dagger_a \gamma^{[ia}{}_b \gamma^{jb}{}_c \gamma^{k]c}{}_d \psi^d, \qquad N^{ijkl} = \psi^\dagger_a \gamma^{[ia}{}_b \gamma^{jb}{}_c \gamma^{kc}{}_d \gamma^{l]d}{}_e \psi^e. \tag{6}$$

This makes it nonessential to introduce a nonholonomic orthogonal coordinate system in order to define the spinor, and makes it possible to define naturally the invariance group of the spinor field ψ^a as the invariance group of the corresponding tensor fields (6)*.

2. Let us find all general solutions of equations (1) and (2) that are invariant with respect to 3-dimensional groups G_3 acting on 2-dimensional nonisotropic transitivity surfaces V_2. The form of the admissible metrics in this case is well known ([6]). If the groups G_3 act on submanifolds V_2 of index 1, then, with a suitable choice of orthoframe, the invariance condition gives $J^4 = 0$, whence $\psi^a = 0$ $(a = 1, 2, 3, 4)$.

In the remaining cases we consider an invariant metric of the form

$$ds^2 = A\,(dx^{1\,2} + s(x^1)dx^{2\,2}) + B\,dx^3dx^4, \tag{7}$$

where $A, B \parallel x^3, x^4,\ A > 0,\ B > 0;\ s = 1)\ \sin^2 x^1,\ 2)\ \mathrm{sh}^2\, x^2,\ 3)\ 1.$

By virtue of the invariance of the model with respect to the choice of orthoframe, it may be chosen as follows:

$$b_1^1 = 1/\sqrt{A}, \qquad b_2^2 = 1/\sqrt{As}, \qquad b_3^3 = b_4^4 = b_3^4 = -b_4^3 = 1/\sqrt{B}. \tag{8}$$

The invariance conditions for the tensors (6) lead to the following nonzero functions of the variables $x^3,\ x^4$: $\Omega,\ J^3,\ J^4,\ M^{12}/\sqrt{s},\ M^{34},\ S^{123}\sqrt{s},\ S^{124}\sqrt{s},\ N^{1234}\sqrt{s}$, whence, taking into account the possible transformation $x^3 \leftrightarrow x^4$, we obtain

$$\psi^a = \delta_2^a \rho_2 e^{i\sigma_2} + \delta_3^a \rho_3 e^{i\sigma_3}, \tag{9}$$

where $\rho_2, \rho_3, \sigma_2 - \sigma_3 \parallel x^3, x^4$.

After computing the operators Γ_{ib}^a (5), the Dirac equation (1) takes the form

$$\rho_3\partial_1\sigma_3 = \rho_2\partial_1\sigma_2 = \rho_3\sigma_3' = \rho_2\dot{\sigma}_2 = 0,$$

$$\rho_3\left(\partial_2\sigma_3 - \frac{1}{4\sqrt{s}}\frac{ds}{dx^1}\right) = \rho_2\left(\partial_2\sigma_2 - \frac{1}{4\sqrt{s}}\frac{ds}{dx^1}\right) = 0,$$

$$2\rho_3' + \frac{A'}{A}\rho_3 + \frac{B'}{2B}\rho_3 = 0, \qquad 2\dot{\rho}_2 + \frac{\dot{A}}{A}\rho_2 + \frac{\dot{B}}{2B}\rho_2 = 0, \tag{10}$$

where $A' = \partial_3 A,\ \dot{A} = \partial_4 A, \ldots$. Hence it follows directly that the first two cases lead to the trivial solution for ψ^a, $\rho_2 = \rho_3 = 0$**.

In the last case, when the group G_3 is the group of motions of the plane, the Einstein equations (2) have the form

$$R_{11} = -2\dot{A}'/B = 0, \qquad R_{33} = -A''/A + {A'}^2/2A^2 + A'B'/AB = \varkappa\sqrt{B}\rho_2^2\sigma_2,$$

$$R_{34} = -\dot{A}'/A + A'\dot{A}/2A^2 - \dot{B}'/B + B'\dot{B}/B^2 = 0, \tag{11}$$

$$R_{44} = -\ddot{A}/A + \dot{A}^2/2A^2 + \dot{A}\dot{B}/AB = -\varkappa\sqrt{B}\rho_3^2\sigma_3.$$

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* The invariance group of the tensor field $T(x)$ is the group of transformations $x' = x'(x)$ for which $T'(x) = T(x')$.

** We note that this result is also valid for the Dirac equation of the form $\gamma^b i a_b^\nu \nabla_i \psi^b + m\psi^a = 0$.

Equations (10), (11) are easily integrated:

$$A = \alpha \int \varphi\, dx^3 - \varkappa \int \varphi \int \frac{g\sigma_2'}{\varphi}\, dx^3 dx^3 + \beta \int \chi\, dx^4 + \varkappa \int \chi \int \frac{f\dot{\sigma}_3}{\chi}\, dx^4 dx^4 + \gamma, \tag{12}$$

$$B = \varphi\chi/\sqrt{A}, \qquad \rho_3^2 A\sqrt{B} = f, \qquad \rho_2^2 A\sqrt{B} = g$$

with arbitrary functions $\varphi, g, \sigma_2 \| x^3$ and $\chi, f, \sigma_3 \| x^4$ and constants α, β, γ.

The tensor T_{ij} and the current vector J^k (6), $\nabla_{kJ}^k = 0$, are

$$T_{ij} = \delta_i^3\delta_j^2\, g\sigma_2'/A - \delta_i^4\delta_j^4\, f\dot{\sigma}_3/A, \qquad J^k = \delta_3^k\, 2f/\varphi\chi\sqrt{A} - \delta_4^k\, 2g/\varphi\chi\sqrt{A}. \tag{13}$$

By a choice of coordinate system the metric can be brought to the form

$$ds^2 = A\,(dx^{12} + dx^{22}) + \frac{1}{\sqrt{A}}\, dx^3 dx^4 \tag{14}$$

with $\varphi = \chi = 1$ in formulas (12), (13).

The independent nonzero components of the curvature tensor in the nonholonomic orthonormal coordinate system will be

$$\tilde{R}_{1212} = -\tilde{R}_{3434} = -A'\dot{A}/A^{3/2}, \qquad \tilde{R}_{1313} = \tilde{R}_{2323} = (-A''A + A'\dot{A} - \ddot{A}A)/2A^{3/2}$$

$$\tilde{R}_{1314} = \tilde{R}_{2324} = (-A'' + \ddot{A})/2A^{1/2}, \qquad \tilde{R}_{1414} = \tilde{R}_{2424} = (-A''A - A'\dot{A} - \ddot{A}A)/2A^{3/2}. \tag{15}$$

From this it is easy to compute the Riemannian curvature of the obtained manifold in two-dimensional directions and to determine the scalar invariants of the curvature tensor. The tensor of conformal curvature in the orthonormal frame (9) has the form

$$\tilde{C}_{1212} = -\tilde{C}_{3434} = -2\tilde{C}_{1313} = -2\tilde{C}_{2323} = 2\tilde{C}_{1414} = 2\tilde{C}_{2424} = -A'\dot{A}/A^{3/2}, \quad (16)$$

the remaining components being zero; and, under the mapping of bivector space onto a three-dimensional metric complex space, it is reduced to Petrov' s first canonical type ([6])*:

$$2C^{+}_{11} = 2C^{+}_{22} = -C^{+}_{33} = -A'\dot{A}/A^{3/2}.$$

Analogously to the definition of the mass of a body of finite radius in the case of spherical symmetry ([8]), the constants α, β can be related to the finiteness of the carriers of the functions $T_{33}A = g\sigma'_2$ and $T_{44}A = f\dot{\sigma}_3$

$$\alpha = -\varkappa \int_{\omega_3} g\sigma'_2 \, dx^3, \qquad \beta = \varkappa \int_{\omega_4} f\dot{\sigma}_3 \, dx^4. \quad (17)$$

They characterize the gravitational field created by a particle in vacuum, or its action on the opposite particle. For $T_{ij} = 0$ we obtain the known asymptotically flat Einstein manifold ([6]).

3. Within the framework of the general solution obtained (12), (13), let us consider the influence of a homogeneous flux of free neutrinos along the axis $z = (x^3 + x^4)/2$ on the geometry of space. Setting in (12) $\alpha = 0$ and $\psi^2 = 0$, we choose the functions f and σ_3 in the coordinate system in which the metric has the form

$$ds^2 = A(x^4)(dx^{12} + dx^{22}) + dx^3 dx^4, \quad (18)$$

so that

$$T_{44} = \varepsilon, \qquad J^3 = 2n \quad (19)$$

or $\rho_3^2 = n$, $\dot{\sigma}_3 = -\varepsilon/n$, where the constants ε and n correspond to the energy density and the particle-number density. The solution (12) gives

$$A = \cos^2 \sqrt{\varkappa\varepsilon/2}\, x^4. \quad (20)$$

It follows from (20) that along the z axis, with the speed of light, there propagates a periodic compression of space toward this axis.

The geodesic equations after the choice of the canonical parameter τ have the form

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* It can be shown that the classification of the algebraic types of the conformal curvature tensor is connected with its invariance with respect to the finite groups $\underline{m} \cdot 2 : \underline{m}$, $\underline{2} : \underline{m}$, $\underline{2}$ (see ([7])) respectively for the first, second, and third types.

$$x^1 = \frac{c_1}{\mu} \operatorname{tg} \mu\tau + c_2, \qquad x^2 = \frac{c_3}{\mu} \operatorname{tg} \mu\tau + c_4,$$

$$x^3 = -\frac{c_1^2 + c_3^2}{\mu} (\operatorname{tg} \mu\tau - \mu\tau) + c_5\tau + c_6, \qquad x^4 = \tau, \tag{21}$$

where $\mu = \sqrt{\varkappa\varepsilon/2}$. Thus, the manifold turns out to be incomplete, since not all geodesic lines continue through the singular points

$$\tau = \frac{1}{\mu}\left(\frac{\pi}{2} + k\pi\right), \qquad k = 0, \pm 1, ...$$

These results make it possible to take rigorous account of the influence on the geometry of space of completely polarized neutrino fluxes arising during the expansion of the Universe in cosmological models. For the energy density of neutrinos in cosmic rays $\varepsilon = 5 \cdot 10^{-14}$ g/sec$^2 \cdot$cm ([9]) we have $1/\mu = 4.5 \cdot 10^{30}$ cm, which characterizes the distance between the points of contraction of space.

4. The group of motions of the plane is isomorphic to the subgroup of the Lorentz group leaving invariant an isotropic vector, which is the stationary subgroup of the origin of coordinates of the invariance group of the solution of equations (1) for a plane neutrino wave in the case of special relativity ($\varkappa = 0$) and has the infinitesimal operators

$$X_1 = (y^4 - y^3)\partial_1 + y^1(\partial_3 + \partial_4), \qquad X_2 = (y^4 - y^3)\partial_2 + y^2(\partial_3 + \partial_4),$$

$$X = y^1\partial_2 - y^2\partial_1 \tag{22}$$

and nonisotropic parabolic surfaces of transitivity $y^{1\,2} + y^{2\,2} + y^{3\,2} - y^{4\,2} = c_1,\ y^3 - y^4 = c_2$.

The only transformation (up to an additive constant) taking the operators of the group of motions of the plane $\partial_1, \partial_2, x^1\partial_2 - x^2\partial_1$ into the operators (24) and reducing the metric (14) in the void for $\alpha = \gamma = 0,\ \beta = 1$ to the Euclidean one will be the transformation

$$x^1 = y^1/(y^3 - y^4), \qquad x^2 = y^2/(y^3 - y^4),$$

$$x^3 = (y^{1\,2} + y^{2\,2} + y^{3\,2} - y^{4\,2})/2(y^3 - y^4), \qquad x^4 = (y^3 - y^4)^2, \tag{23}$$

which has a singularity at $x^4 = 0$ and, consequently, changes the global properties of the manifold. The hypersurfaces $x^3 = c$ now represent cones $y^{1\,2} + y^{2\,2} + (y^3 - c)^2 - (y^4 - c)^2 = 0$ with common straight line $y^1 = y^2 = y^3 - y^4 = 0$, i.e., a family of contracting and then expanding spheres S_2 in the space of variables y^1, y^2, y^3, with centers located along the y^3 axis and with a common point.

Thus, in this interpretation, for a small carrier ω_3 of the function $g\sigma_2'$ (see (17)), we obtain an almost spherical perturbation of the antineutrino energy density.

For $\psi^2 = 0$, from (12), (14) we have

$$A = \alpha x^3 + x^4 + \varkappa \iint f\sigma_3 \, dx^4 dx^4 \tag{24}$$

and the curvature tensor (15) tends to zero as $y^{1\,2} + y^{2\,2} \to \infty$.

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REFERENCES

1. D. Brill, J. Wheeler, Rev. Mod. Phys., **29**, 465 (1957); Russian translation in the collection *Newest Problems of Gravitation*, IL, 1965.
2. L. I. Sedov, UMN, **20**, no. 5 (1965).
3. A. N. Golubyatnikov, DAN, **169**, no. 2 (1966).
4. H. Weyl, Zs. Phys., **56**, 330 (1929).
5. V. A. Zhelnorovich, PMM, **30**, no. 6 (1966).
6. A. Z. Petrov, *New Methods in General Relativity*, "Nauka," 1966.
7. A. N. Golubyatnikov, V. V. Lokhin, DAN, **187**, no. 2 (1969).
8. L. D. Landau, E. M. Lifshitz, *Field Theory*, "Nauka," 1967.
9. J. Wheeler, *Gravitation, Neutrinos, and the Universe*, IL, 1962.