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Abstract

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ON A SEQUENCE OF LINEAR POSITIVE OPERATORS

In this article a sequence of linear positive operators of a very general kind is constructed, containing as special cases the well-studied operators of S. N. Bernstein, Bernstein–Chlodovsky, G. Mirakyan, V. A. Baskakov, and others. Questions of uniform convergence of these operators in the class of continuous functions $C[0, A]$, where $A > 0$ is a given number, are studied, as well as their properties of convexity or concavity.

I. Consider two sequences of functions $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ from the class $C[0, A]$ such that

$$\varphi_n(0) = 0, \quad \psi_n(0) \neq 0, \quad \psi_n(t) > 0 \quad (0 \leq t \leq A, \ n = 1, 2, \ldots), \tag{1}$$

and let $\{a_n\}$ be a sequence of positive numbers having the properties

$$\lim_{n\to\infty} a_n = \infty, \quad \lim_{n\to\infty} \frac{\alpha_n}{n} = 1, \quad \lim_{n\to\infty} \frac{1}{n^2 \psi_n(0)} = 0. \tag{2}$$

Suppose that the sequence of functions of three variables $\{K_n(x, t, u)\}$ $(x, t \in [0, A], \ -\infty < u < \infty)$ satisfies the following conditions:

1°. Each function of this sequence is an entire analytic function with respect to u for fixed x and t from the interval $[0, A]$.

2°. $K_n(x, 0, 0) = 1$ $(n = 1, 2, \ldots)$ for any $x \in [0, A]$.

3°.

$$\left\{(-1)^\nu \left[\frac{\partial^\nu}{\partial u^\nu} K_n(x, t, u)\right]_{t=0}^{u=u_1}\right\} \geq 0 \quad * \quad (\nu, n = 1, 2, \ldots; \ x \in [0, A]).$$

4°.

$$-\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{\substack{u=u_1\\ t=0}} = nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{n+m}(x,t,u)\right]\Big|_{\substack{u=u_1\\ t=0}} \qquad (\nu, n =$$

$$= 1,2,\dots,\ x\in[0,A]),$$

where m is a natural number.

Consider the sequence of linear positive operators

$$L_n[f;x]=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\left\{\left[\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\right]_{t=0}^{u=\alpha_n\psi_n(0)}\right\}\frac{[-\alpha_n\psi_n(0)]^{\nu}}{\nu!}. \quad (3)$$

It is obvious that by applying property 4° ν times the operator (3) can be brought to the form

$$L_n[f;x]=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{n(n+m)\dots[n+(\nu-1)m]}{\nu!}\times$$

$$\times[\alpha_n\psi_n'(0)]^{\nu}K_{n+\nu m}(x,0,\alpha_n\psi_n(0))\,x^{\nu}. \quad (4)$$

II. We shall show that the operator $L_n[f;x]$ contains as a special case a number of known operators.

1. In the case $K_n(x,t,u)=[1-xu/(1+t)]^n$, the operator $L_n[f;x]$ takes the form

$$L_n^{\langle 1\rangle}[f;x]=\sum_{\nu=0}^{n} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\binom{n}{\nu}[1-x\alpha_n\psi_n(0)]^{n-\nu}(\alpha_n\psi_n(0)x)^{\nu}.$$

The conditions 1°–4°, obviously, are satisfied, and $m=1$. For $\alpha_n=n$, $\psi_n(0)=1/n$, the operator $L_n^{\langle 1\rangle}[f;x]$ coincides with the classical polynomial of S. N. Bernstein.

Moreover, putting

$$\alpha_n=n, \qquad \psi_n(0)=\frac{1}{nb_n}\left(\lim_{n\to\infty}b_n=\infty,\ \lim_{n\to\infty}\frac{b_n}{n}=0\right),$$

we arrive at the Bernstein–Chlodowsky polynomials

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* This notation means that the derivative with respect to u is taken ν times, and then $u=u_1$ and $t=0$ are set.

$$L_n^{\langle 2\rangle}[f;x] = \sum_{\nu=0}^{n} f\left(\frac{\nu b_n}{n}\right)\binom{n}{\nu}\left(1-\frac{x}{b_n}\right)^{n-\nu}\left(\frac{n}{b_n}\right)^{\nu}.$$

2. Let us now put $K_n(x,t,u) = e^{-n(t+xu)}$. The conditions 1°–4°, as is easily verified, are satisfied, and $m = 0$. In this case $L_n[f;x]$ takes the form

$$L_n^{\langle 3\rangle}[f;x] = e^{-nx\alpha_n\psi_n(0)}\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{(nx)^\nu}{\nu!}(\alpha_n\psi_n(0))^\nu.$$

For $\alpha_n = n$, $\psi_n(0) = 1/n$, this operator coincides with the operator of G. M. Mirakyan.

3. Let $K_n(x,t,u) = K_n(t+xu)$, where $K_n(z)$ is an entire analytic function. Simple computations show that in this case $L_n[f;x]$ takes the form

$$L_n^{\langle 4\rangle}[f;x] = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{1}{\nu!}K_n^{(\nu)}(x\alpha_n\psi_n(0))[-x\alpha_n\psi_n(0)]^\nu.$$

For $\alpha_n = n$, $\psi_n(0) = 1/n$, we arrive at the operators of V. A. Baskakov (^{1}). If, however, one puts $\alpha_n = n$, $\psi_n(0) = 1/\alpha_n$ and denotes $n^2/\alpha_n = \beta_n$, then the operator $L_n^{\langle 4\rangle}[f;x]$ turns into another operator of V. A. Baskakov (^{2}).

III. With regard to the convergence of the operators $L_n[f;x]$ in the class $C[0,A]$, where $A > 0$ is a given number, the following assertions hold.

Theorem 1. *If the function $f(x)$ from the class $C[0,A]$ is continuous from the right at the point $x = A$ and grows at infinity no faster than x^2, then, uniformly on $[0,A]$, we have*

$$\lim_{n\to\infty} L_n[f;x] = f(x).$$

Proof. Obviously, it is sufficient to verify the conditions of the well-known theorem of P. P. Korovkin (the growth restrictions on the function, as indicated in (^{2}), do not change its proof).

Writing for the function $K_n(x,t,u)$ the Taylor expansion in powers of $(u-u_1)$, we put in it $u = \varphi_n(t)$, $u_1 = \alpha_n\psi_n(t)$, where $\varphi_n(t)$ and $\psi_n(t)$ and the sequence $\{\alpha_n\}$ are defined in (1) and (2). Then putting $t = 0$, by virtue of condition 2°, we obtain $L_n[1;x] = 1$.

Next, according to 3°, we have

$$L_n[y;x] = \frac{x\alpha_n}{n}\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_{n+m}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{\nu!} = \frac{x\alpha_n}{n},$$

and, consequently, by virtue of (2), $L_n[y;x] \rightrightarrows x$ $(n \to \infty,\ x \in [0, A])$. Finally, applying condition 3° twice, we obtain

$$L_n[y^2;x] = \left(\frac{x\alpha_n}{n}\right)^2 \frac{n+m}{n} \sum_{\nu=2}^{\infty} \frac{\partial^{\nu-2}}{\partial u^{\nu-2}} K_{n+2m}(x, 0, \alpha_n\psi_n(0)) \frac{[-\alpha_n\psi_n(0)]^{\nu-2}}{(\nu-2)!} +$$

$$+\frac{x\alpha_n}{n}\frac{1}{n^2\psi_n(0)} = \left(\frac{x\alpha_n}{n}\right)^2 \frac{n+m}{n} + \frac{x\alpha_n}{n}\frac{1}{n^2\psi_n(0)};$$

by virtue of (2) it follows from this that, for $x \in [0, A]$, $L_n(y^2;x) \rightrightarrows x^2$ $(n \to \infty)$. The theorem is proved.

A more general result is contained in the following theorem.

Theorem 2. *If the function $K_n(x, t, u)$, in addition to conditions 1°–3°, also satisfies the condition*

$$\lim_{n\to\infty} \frac{1}{n}\frac{\partial}{\partial u} K_n(x, 0, 0) = -x, \qquad \lim_{n\to\infty} \frac{1}{n^2}\frac{\partial^2}{\partial u^2} K_n(x, 0, 0) = x^2$$

uniformly with respect to $x \in [0, A]$, then the sequence of operators $L_n[f;x]$ converges uniformly on $[0, A]$ to the function $f(x) \in C[0, A]$, continuous from the right at the point $x = A$ and increasing at infinity no faster than x^2.

The proof can be carried out by the same method as in (2). We note that Theorem 1 can be obtained from Theorem 2.

IV. We now show that the operators $L_n[f;x]$ preserve convexity, concavity, and polynomiality of arbitrary order on the interval $[0, A]$. As is known (3), a real function $f(x)$ is called convex, non-concave, polynomial, non-convex, concave of k-th order on $[a, b]$, if its divided difference of order $(k+2)$, $[x_1, x_2, \dots, x_{k+2}; f]$, for any system of $(k+2)$ points of the interval $[a, b]$, is respectively $> 0,\ \geq 0,\ = 0,\ \leq 0,\ < 0$. For functions $F(x) \in C^{(k+1)}[0, A]$, as T. Popoviciu showed (see, for example, (3)), the conditions of convexity, polynomiality, and concavity of k-th order on $[0, A]$ mean that on $[0, A]$ there hold, respectively, the relations

$$F^{(k+1)}(x) > 0,\ = 0,\ < 0.$$

Let us now suppose that the function $K_n(x, t, u)$, besides conditions 1°–4°, also satisfies the condition

$$-\frac{\partial}{\partial x} K_n(x, t, u)\Big|_{\substack{u=u_1\\ t=0}} = nu_1 K_{n+m}(x, 0, u_1), \tag{5}$$

where m is the natural number defined in 4°.

Differentiating (4) with respect to x and taking (5) into account, by induction one can obtain the equality

$$\frac{d^p}{dx^p} L_n[f;x] = \frac{\alpha_n^p p!}{n^{2p}} \sum_{\nu=0}^{\infty} \left[\frac{\nu}{n^2\psi_n(0)}, \dots, \frac{\nu+p}{n^2\psi_n(0)}; f\right] \times$$

$$\times K_{n+(\nu-p)m}(x, 0, \alpha_n\psi_n(0)) \frac{[\alpha_n\psi_n(0)]^\nu}{\nu!} n(n+m)\cdots[n+(\nu+p-1)m]\, x^\nu,$$

where $[x_1, x_2, \dots, x_p; f]$ is the divided difference of the function $f(x)$ at the nodes $x_1 = \nu/n^2\psi_n(0), \dots, x_p = (\nu+p)/n^2\psi_n(0)$. With the help of this representation one obtains

Theorem 3. *If the function $f(x)$ is convex (concave) or polynomial of k-th order on the half-axis $[0, \infty)$, then the sequence of operators $L_n[f;x]$ on the interval $[0, A]$ is respectively convex (concave) or polynomial.*

In conclusion, we note that the operator $L_n[f;x]$ can be represented in the form

$$L_n[f;x] = \sum_{\nu=0}^{\infty} \frac{\nu!}{n^{2\nu}} n(n+m)\dots[n+(\nu-1)m](\bar{a}_n x)^\nu \times$$

$$\times \left[0, -\frac{1}{n^2\psi_n^*(0)}, \dots, -\frac{\nu}{n^2\psi_n^*(0)}; f\right] K_{n+\nu m}(0, 0, a_n'\psi_n^*(l)).$$

From this representation it follows that if $f(x)$ is a polynomial of degree r, then $L_n[f;x]$ is its polynomial of degree $\leq r$, since in this case all divided differences of the function $f(x)$ of order greater than r are equal to zero.

We note that similar assertions for the operator of V. A. Baskakov (^{1}) were proved in (^{3}).

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[1] V. A. Baskakov, DAN, **113**, 249 (1957).
[2] V. A. Baskakov, *Studies on Contemporary Problems of the Constructive Theory of Functions*, Moscow, 1961.
[3] A. Lupas, Mathematica, **9** (32), I, 77 (1967); **11**, 295 (1967).