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DIFFERENTIAL GAMES OF APPROACH WITH CONSTRAINED PHASE COORDINATES

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In accordance with the approach ([1,2]), formalized and approximate solutions of differential games of approach ([3]) are considered for constrained coordinates and in the presence of lags in the system.

1. Programmed approach. On the interval $[t_\alpha, \vartheta]$ there is given an n-dimensional controlled system

$$dx/dt = A(t)x + B(t)u - C(t)v \tag{1}$$

with continuous matrix coefficients and, respectively, with r- and q-dimensional controls u, v of the 1st and 2nd players. Constraints are prescribed: $u(t) \in U(t)$, $v(t) \in V(t)$, $\{x\}_k \in X_k(t)$, where, for every t, $U(t), V(t), X_k(t)$ are convex compact sets in E_r, E_q, E_k, continuous in t; $\{x\}_k$ is the k-vector $\{x_1, \dots, x_k\}$, where x_j are the coordinates of x. Let M be a given bounded convex set in the space E_m of selected coordinates x_{j_s}, $s = 1, \dots, m$, $\{x\}_m = \{x_{j_1}, \dots, x_{j_m}\}$, and let $r[\{x\}_m, M]$ be the Euclidean distance from $\{x\}_m$ to M.

Problem 1 (on programmed approach). For a given initial position t_α, $x_\alpha = x(t_\alpha)$, find controls $u^0(t), v^0(t)$ which, by virtue of (1), deliver the condition

$$\varepsilon^0[x_\alpha, t_\alpha] = \varepsilon(u_0, v_0) = \max_{v \in V} \min_{u \in U} \varepsilon(u, v) = \max_{v \in V} \min_{u \in U} r[\{x(\vartheta)\}_m, M].$$

Problem 1* **(adjoint).** Among first-order distributions of the form $\{\lambda = l_\alpha \delta(t - t_\alpha) + l_\beta \delta(t - t_\beta) - l(t)\} = \{\lambda\}$ ($l_\alpha = \{l_{\alpha j}\}$, $l_\beta = \{l_{\beta j}\}$, $l(t) = \{l_j(t)\}$, $j = 1, \dots, n$; $l_{\beta j} = 0$, if $j \neq j_s$; $l(t) \equiv 0$, if $j > k$), and such that the corresponding solution $s_\lambda(t)$ of the adjoint equation in distributions $ds/dt = -sA + \lambda$ is concentrated ([4]) on $[t_\alpha, \vartheta]$, find an optimal λ^0, delivering the maximum of the functional

$$\Psi(\lambda^0) = \max_{l,l_\beta} \Psi[l, l_\beta] = \max_{l,l_\beta} \Big\{ \rho^{(2)}_{t_\alpha}[sC] - \rho^{(1)}[sB] -$$

$$- \int_{t_\alpha}^{t_\beta} \gamma_X[\{d\Lambda\}_k] - l_\alpha x_\alpha - \gamma_M[\{l_\beta\}_m] \Big\} \tag{2}$$

under the condition

$$\|l_\beta\|^2 = \sum_{j=1}^{n} l^2_{\beta j} = 1.$$

Here $l(t) = d\Lambda(t)/dt$ (in the generalized sense), $\rho^{(1)}_{t_\alpha}[h]$, $\rho^{(2)}_{t_\alpha}[h]$ are the support functionals of the sets $\{u(t)\} \subset U(t)$ and $\{v(t)\} \subset V(t)$, $t_\alpha \le t \le \vartheta$; $\gamma_X[p]$, $\gamma_M[p]$ are the support functions of the sets $-X_k$ and M. Denote $s_\lambda(t)|_{\lambda=\lambda^0} = s^0(t)$, $\lambda^0 = l^0_\alpha \delta(t - t_\alpha) + l^0_\beta \delta(t - t_\beta) - l^0(t)$, $d\Lambda^0/dt = l^0(t)$.

Theorem 1. The solutions $\varepsilon^0[x, t]$, $u^0(t)$, $v^0(t)$, $x^0(t)$ of Problem 1 satisfy, respectively, the condition $\varepsilon^0[x, t] = \Psi[l^0, l^0_\beta]$ and the maximum principle

for the controls and trajectory

$$s^0(t)B(t)u^0(t) = \max_{u \in U(t)} s^0(t)B(t)u, \quad s^0(t)C(t)v^0(t) = \max_{v \in V(t)} s^0(t)C(t)v; \tag{3}$$

$$\int_{t_\alpha}^{\vartheta} d\Lambda^0(t)x^0(t) = -\max_p \int_{t_\alpha}^{\vartheta} d\Lambda^0(t)p(t), \tag{4}$$

where the maximum is taken over all k-vector functions $p(t)$ satisfying the inclusion $p(t) \in -X_k(t)$.

2. Differential game of approach. For system (1) consider the differential game in which the payoff is taken to be $\chi[\vartheta] = r[\{x(\vartheta)\}_m, M]$. Let $X(t) = \{x : \{x\}_k \in X_k(t)\}$, $T = [t_\alpha, \vartheta]$.

Assumption 1. One can specify $\delta > 0$ such that, for any $(n+1)$-vector $\{x, t\}$ from $D = X \times T$, the set $\{u(\xi)\}$ of controls $u(\xi) \in U(\xi)$ preserving, for $t \le \xi \le t + \delta$, the inclusion $x(\xi) \in X(\xi)$, whatever $v(\xi) \in V(\xi)$ may be, is nonempty.

Write $D = D_i \cup D_s \cup D_n$, where $D_i = \operatorname{int} X \times T$, $D_s \cup D_n = \partial X \times T$. An admissible strategy $\mathcal{U}(x, t)$ ($\mathcal{V}(x, t)$) for the position $\{x, t\}$ will mean a multivalued function $\mathcal{U}(x, t) \subset U(t)$ ($\mathcal{V}(x, t) \subset V(t)$) with values in a convex compact set, upper semicontinuous with respect to inclusion in $D_i \cup D_n$ (in D). Let, in the domain D_s, $\mathcal{U}(x,t)$ coincide with the set of vectors $\{u = u(t)\}$ corresponding to the functions $\{u(\xi)\}$ of Assumption 1, and, in the domain D_n, let $\mathcal{U}(x,t) \subset \{u = u(t)\}$. Substituting a pair of admissible strategies into (1), we obtain the equation in contingencies (5), or the generalized dynamical system (6),

$$dx/dt \in A(t)x + B(t)\mathcal{U}(x,t) - C(t)\mathcal{V}(x,t). \quad (5)$$

The set $\{x(t)\}$ of solutions of (5) (i.e., absolutely continuous functions satisfying the inclusion (5) almost everywhere) is nonempty; each solution $x(t)$ everywhere satisfies the inclusion $x(t) \in X(t)$ and is generated, by virtue of (1), by a pair of measurable functions $u(t) \in \mathcal{U}(x,t)$, $v(t) \in \mathcal{V}(x,t)$. Let $\sigma[u,v,t,x] = (\chi[x(\vartheta)]/u(t), v(t), t, x)$.

Problem 2. Find optimal game strategies ensuring the condition

$$\sigma^0 = \min_{U} \max_{v(t)} \inf_{\{x(t)\}} \sigma[\mathcal{U}, v, t, x] = \max_{V} \min_{u(t)} \sup_{\{x(t)\}} \sigma[u, \mathcal{V}, t, x]. \quad (6)$$

Assumption 2. For each position $\{x,t\} \in D$, the corresponding problem 1^*, considered on $[t,\vartheta]$, has a unique solution—an extremal distribution λ^0. The function $\Lambda^0(t)$ contains no singular component.

The condition for $\Lambda^0(t)$ is discussed in (7). Assumption 2 is certainly satisfied if $B(t) \equiv C(t)$ and the sets $X_k(t)$ are uniformly convex uniformly in t.

Theorem 2. Suppose Assumptions 1 and 2 are satisfied. Then there exist optimal strategies $\mathcal{U}^0(x,t)$, $\mathcal{V}^0(x,t)$ solving Problem 2.

For $\{x,t\} \subset D_i \cup D_n$, $\mathcal{U}^0(x,t)$, $\mathcal{V}^0(x,t)$ coincide with the sets of vectors u, v satisfying, at time t, the maximum principle (3) for Problem 1 for $\{x,t\}$ and $[t,\vartheta]$. Here D_n coincides with the set of vectors $\{x,t\} : x \in \partial X$ for which, in the indicated Problem 1, we have $s^0(t)B(t) \neq 0$. Then the closed domain D_s is the complement of $D_i \cup D_n$ in D. In D_s, $\mathcal{U}^0(x,t)$ is defined as an admissible strategy and $\mathcal{V}^0(x,t)$ is defined in the same way as in $D_i \cup D_n$. The proof of Theorem 2 relies, in particular, on the property of weak continuity of the operator defined on $D \subset E_{n+1}$, $(\lambda^0(\xi)/x,t)$, $t \le \xi \le \vartheta$, from E_n into $D'[t,\vartheta]$, and on the resulting admissibility of the strategies $\mathcal{U}^0(x,t)$, $\mathcal{V}^0(x,t)$. Under the conditions of Theorem 2, the function $\varepsilon^0[x,t]$ in the domain $\varepsilon^0[x,t] > 0$ is right-differentiable along the realizing motion $x[t]$, and

$$(d\varepsilon^0[x,t]/dt)_{+0} = \partial(\rho_t^{(2)}[s^0C] - \rho_t^{(1)}[s^0B])/\partial t - d(s^0(t)x[t])/dt + \max_{x\in -X(t)} l^0(t)x = s^0(t)\,[B(t)(u^0(t)-u(t)) - C(t)(v^0($$

The solution of Problem 2 can be realized in approximative strategies. Indeed, dividing $[t_\alpha,\vartheta]$ into N equal parts $\{t_\alpha = t_0, \dots, \dots, t_N = \vartheta\}$, $t_{i+1} - t_i = \Delta$, we can, relying on Problem 1, define at each step $t_i \le t < t_{i+1}$ the control $u_\Delta[t] = u[x_\Delta[t_i], t_i] \in \mathcal{U}^0_\zeta(x_\Delta[t_i], t_i)$, where $x_\Delta[t]$ is the solution of (1) for $u = u_\Delta[t]$, and $\mathcal{U}_\zeta$ is the Euclidean ζ-neighborhood of $\mathcal{U}$, $\zeta(\Delta) \to 0$ $(\Delta \to 0)$.

Theorem 3. Suppose assumptions 1, 2 are satisfied. Then, whatever the numbers $\delta > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and the realization $v(\xi) \subset V(\xi)$, for every bounded domain $G(t) \subset X(t) \subset E_n$ one can specify a number Δ_0 and a control u_Δ such that, for $\Delta \le \Delta_0$, the inequalities

$$\sigma[u^0_\Delta[\xi], v(\xi), t, x] - \sigma^0 \le \delta, \quad r[x^0[\xi], X(\xi)] \le \delta_1, \quad r[u^0_\Delta[\xi], U(\xi)] \le \delta_2,$$

hold for $t \le \xi \le \vartheta$, whatever the initial position $\{x, t\}$, where $x \in G(t)$, $t_\alpha \le t \le \vartheta$.

In an analogous manner the strategy $\mathcal{V}^0(x,t)$ is approximated for any realization $u(\xi) \subset U(\xi)$; approximative solutions and strategies of Problem 2 with the saddle point of the corresponding game are constructed.

3. Game approach of systems with delay. On the interval $[t_\alpha, \vartheta]$ consider two n-dimensional systems with constant delays and continuous matrix coefficients

$$dy/dt = A(t)y + A_h(t)y(t-h) + B(t)u,$$
$$dz/dt = C(t)z + C_\tau(t)z(t-\tau) + D(t)v, \tag{7}$$

controlled respectively by r- and q-vector functions $u(t) \in U(t)$, $v(t) \in V(t)$, under the conditions $\{y\}_k \in Y_k(t)$, $\{z\}_l \in Z_l(t)$, where $\{z\}_l = \{z_{r_1}, \dots, \dots, z_{r_s}, \dots, z_{r_l}\}$ and $Y_k(t)$, $Z_l(t)$ are convex compact sets in E_k, E_l, continuous in t. Here the solution of the program problem of finding, for a given position $\{y_{t_\alpha}(\zeta), z_{t_\alpha}(\eta), t\}$, $-h \le \zeta \le 0$, $-\tau \le \eta \le 0$, $y_t(\zeta) = y(t+\zeta) \in C[-h, 0]$, $z_t(\eta) = z(t+\eta) \in C[-\tau, 0]$, controls $u^0(t)$, $v^0(t)$ satisfying the condition

$$\omega^0[y_{t_\alpha}(\zeta), z_{t_\alpha}(\eta), t_\alpha] = \omega(u^0, v^0) = \max_{v\in V}\min_{u\in U}\omega(u,v) =$$
$$= \max_{v\in V}\min_{u\in U}\|\{y(\vartheta) - z(\vartheta)\}_m\|,$$

leads to the maximum principle

$$s^0_u(t)B(t)u^0(t) = \max_{u\in U(t)} s^0_u(t)B(t)u, \qquad s^0_v(t)D(t)v^0(t) = \max_{v\in V(t)} s^0_v(t)D(t)v, \tag{8}$$

where $s^0_u(t)$, $s^0_v(t)$ are solutions, concentrated on $[t_\alpha, \vartheta]$, of the adjoint systems in distributions

$$ds_u(t)/dt = -s_u(t)A(t) - s_u(t+h)A_h(t+h) + \lambda_u(t),$$
$$ds_v(t)/dt = -s_v(t)C(t) - s_v(t+\tau)C_\tau(t+\tau) + \lambda_v(t), \tag{9}$$

generated by distributions λ^0_u, $\lambda^0_v \in \{\lambda\}$, optimizing the functional

$$\omega[y_{t_\alpha}(\zeta), z_{t_\alpha}(\eta), t_\alpha] = \max_{\lambda_u,\lambda_v}\{\rho^{(2)}_{t_\alpha}[s_{vD}] - \rho^{(1)}_{t_\alpha}[s_{uB}] + l_{\alpha v}z_\alpha - l_{\alpha u}y_\alpha +$$

$$+\int_{-\tau}^{0} s_v(\eta+\tau+t_\alpha)C_\tau(\eta+\tau+t_\alpha)z_{t_\alpha}(\eta)\,d\eta-\int_{-h}^{0} s_u(\xi+h+t_\alpha)A_h(\xi+h+$$

$$+t_\alpha)y_{t_\alpha}(\xi)\,d\xi+\int_{t_\alpha}^{\vartheta}\gamma_y[\{d\Lambda_v\}_l]-\int_{t_\alpha}^{\vartheta}\gamma_z[\{d\Lambda_u\}_k]\},\qquad l_{iu}=l_{\beta v}=l_\beta,$$

$$\|l_\beta\|=1;\qquad l_{uj}(t)=0,\ j\geq k;\qquad l_{vj}(t)=0,\ j\neq r_s;\qquad l_{\beta j}=0,\ j\neq j_s. \quad (10)$$

Passing to a differential game with payoff $\chi[y(\vartheta),z(\vartheta)]=\|\{y(\vartheta)-z(\vartheta)\}_m\|$, by admissible strategies of the players we mean set-valued functionals $\mathcal{U}(y_t(\zeta),t)=\mathcal{U}(\cdot,t)\subset U(t)$, $\mathcal{V}(z_t(\eta),t)=\mathcal{V}(\cdot,t)\subset V(t)$, with values in convex compact sets, upper semicontinuous with respect to inclusion in $\{y_t(\zeta),t\}$ and $\{z_t(\eta),t\}$, respectively, in the domains $\{y(t),t\}\in D_i^{(1)}=\operatorname{int}Y\times T$, $\{z(t),t\}\in D_i^{(2)}=\operatorname{int}Z\times T$. In the domains $\{y(t),t\}\in D_n^{(1)}\cup D_s^{(1)}=\partial Y\times T$, $\{z(t),t\}\in D_n^{(2)}\cup D_s^{(2)}=\partial Z\times T$, the strategies are defined according to the same scheme as in Sec. 2, proceeding from the maximum principle (8), taking into account assumptions analogous to 1, but now holding for each of the systems (7). The case in which the indicated assumptions are satisfied, when for each position $\{y_t(\zeta),z_t(\eta),t\}$ the distributions λ_u^0,λ_v^0 of problem (10) on $[t,\vartheta]$ are unique and Λ_u^0,Λ_v^0 contain no singular components, will be called regular. Substituting the admissible strategy $\mathcal{U}(\cdot,t)$ into (7), we obtain for the first player the equation in contingencies with delay

$$dy(t)/dt\in A(t)y(t)+A_h(t)y(t-h)+B(t)\mathcal{U}(y_t(\zeta),t),$$

whose set $\{y(t)\}$ of solutions is nonempty; moreover everywhere $y(t)\in Y(t)$, and every $y(t)$ is realized by a measurable function $u(t)\in\mathcal{U}(\cdot,t)$. The same conclusion is also valid for the set of solutions $\{z(t)\}$ of the equation in contingencies for the second player.

Theorem 4. *In the regular case of the game of approach of systems (7), there exist optimal strategies $\mathcal{U}^0(\cdot,t)$, $\mathcal{V}^0(\cdot,t)$ satisfying the condition*

$$\chi^0=\min_{U}\max_{v(t)}\inf_{\{y(t)\}}\chi[\mathcal{U},v,t,\cdot]=\max_{V}\min_{u(t)}\sup_{\{z(t)\}}\chi[u,\mathcal{V},t,\cdot],$$

where

$$\chi[u,v,t,\cdot]=(\|\{y(\vartheta)-z(\vartheta)\}_m\|/u(t),v(t),t,y_t(\zeta),z_t(\eta)).$$

The construction of the strategies $\mathcal{U}^0,\mathcal{V}^0$ here is analogous to Sec. 2. The theorems on an approximate solution of the game problem of approach of systems (7) are formulated in an analogous way.

Remark 1. The methods of Secs. 1–3 prove effective in considering the game problem of approach of systems (7) with payoff

$$\chi = \max_{\xi} \left(\|y(\vartheta + \xi) - z(\vartheta + \xi)\|_m \right), \qquad -\tau = -h \le \xi \le 0.$$

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1. N. N. Krasovskii, Differentsial' n. uravn., **5**, No. 3, 407 (1969).
2. N. N. Krasovskii, PMM, **32**, No. 5, 793 (1968).
3. R. Isaacs, *Differential Games*, Moscow, 1967.
4. G. E. Shilov, *Mathematical Analysis. Second Special Course*, Moscow, 1965.
5. S. T. Zaremba, Bull. Sci. Math. (2) **60**, 139 (1936).
6. E. Roxin, J. Diff. Equations, **1**, No. 2, 188 (1965).
7. A. B. Kurzhanskii, Yu. S. Osipov, PMM, **33**, No. 4, 705 (1969).

Note: Figure translations are in progress. See original paper for figures.