ON THE APPROXIMATION OF POLYNOMIALS OF TWO VARIABLES BY FUNCTIONS OF THE FORM \(\varphi(x)+\psi(y)\)

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Abstract

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ON THE APPROXIMATION OF POLYNOMIALS OF TWO VARIABLES BY FUNCTIONS OF THE FORM $\varphi(x) + \psi(y)$

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In the present work, for a certain class of functions of two variables, a simple method is proposed for computing the values of the best approximation by functions of the form $\varphi(x) + \psi(y)$; a method is established for finding the best approximating function for polynomials of two variables, by means of which the best approximating function is found.

Let $f(x, y)$ be a function continuous on the rectangle $Q = [a, b; c, d]$, and

$$E[f, \varphi + \psi; Q]_C = E[f] = \inf_{\varphi+\psi} \|f - (\varphi + \varphi)\|, \qquad \|f\| = \sup_{x,y \in Q} |f|.$$

Denote by Π the class of functions $f(x, y)$ having the following property: for arbitrary $x'' \geq x'$, $y'' \geq y'$ from Q, the inequality

$$f(x'', y'') + f(x', y') \geq f(x'', y') + f(x', y'')$$

holds.

Let $\Pi\_$ be the class of functions $f(x, y)$ for which $(-f) \in \Pi$. It is not difficult to understand that if $f_{xy} \geq 0$, then $f \in \Pi$. Indeed, for arbitrary $x'' \geq x'$, $y'' \geq y'$ we have

$$0 \leq \int_{x'}^{x''} \int_{y'}^{y''} f_{xy}\, dx\, dy = f(x'', y'') + f(x', y') - f(x'', y') - f(x', y'').$$

The converse, in general, is not true, since the function $f(x, y)$ need not be differentiable.

Similarly, if $f_{xy} \leq 0$, then $f \in \Pi\_$.

Theorem 1. For an arbitrary continuous function $f(x, y) \in \Pi$,

$$E[f, \varphi + \psi; Q]_C = \frac{1}{4}[f(b, d) + f(a, c) - f(a, d) - f(b, c)].$$

This result, which is a strengthening of a theorem of T. J. Rivlin and R. J. Sibner ([1]) (the case $f_{xy} \geq 0,\ Q = [0, 1; 0, 1]$), also differs in the method of proof.

We also note the relatively complicated methods ([2–4]) for determining the value of the best approximation in the general case.

Corollary 1. For an arbitrary continuous function $f(x, y) \in \Pi_-$,

$$E[f, \varphi + \psi; Q]_C = -\frac{1}{4}[f(b, d) + f(a, c) - f(a, d) - f(b, c)].$$

Corollary 2. If the mixed derivative f_{xy} preserves its sign on the rectangle Q, then

$$E[f, \varphi + \psi; Q]_C = \frac{1}{4}|f(b, d) + f(a, c) - f(a, d) - f(b, c)|.$$

Corollary 3. Let $f = x^\alpha y^\beta g(x, y)$, where $\alpha, \beta \geq 0$ are real numbers and $Q = [0, b; 0, d]$.

Then:

a) if $f \in \Pi$, then $E[f, \varphi + \psi, Q]_C = \frac{1}{4} f(b, d)$;

b) if $f \in \Pi_-$, then $E[f, \varphi + \psi, Q]_C = -\frac{1}{4} f(b, d)$.

Corollary 4. Let $S = [0, 1;\ 0, 1]$. Then for arbitrary real $\alpha, \beta \geq 0$,

$$E[x^\alpha y^\beta, \varphi + \psi;\ S]_C = E[xy, \varphi + \psi;\ S]_C = \frac{1}{4}.$$

This follows immediately from Corollary 3 and from the fact that $x^\alpha y^\beta \in \Pi$. Consider the approximation of the polynomial in two variables

$$z = \sum_{p=0}^{m} \sum_{q=0}^{n} A_{pq} x^p y^q$$

on the rectangle Q by all possible functions of the form $\varphi(x) + \psi(y)$ in the metric of the space C. Taking into account

$$\sum_{p=0}^{m} \sum_{q=0}^{n} A_{pq} x^p y^q = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} x^p y^q + \sum_{p=0}^{m} A_{p0} x^p + \sum_{q=1}^{n} A_{0q} y^q,$$

we conclude that

$$E\left[\sum_{p=0}^{m}\sum_{q=0}^{n}A_{pq}x^p y^q, \varphi+\psi\right] = E\left[\sum_{p=1}^{m}\sum_{q=1}^{n}A_{pq}x^p y^q, \varphi+\psi\right],$$

and if $\varphi_0+\psi_0$ is the best approximant to the polynomial $\sum_0^m\sum_0^n A_{pq}x^p y^q$, then

$$\varphi_0+\psi_0-\left(\sum_{p=0}^{m}A_{p0}x^p+\sum_{q=1}^{n}A_{0q}y^q\right)$$

is the best approximant to the polynomial $\sum_1^m\sum_1^n A_{pq}x^p y^q$.

Therefore, in what follows we shall assume

$$z=\sum_1^m\sum_1^n A_{pq}x^p y^q.$$

We propose the following method α for determining the best approximating function:

$$z=\sum_1^m\sum_1^n A_{pq}x^p y^q,$$

$$z_1=z, \qquad z_{2n}=z_{2n-1}-g_n, \qquad z_{2n+1}=z_{2n}+h_n,$$

where

$$g_n=g_n(x)=\frac{1}{2}\left[\overset{\wedge}{\max_{c\le y\le d}} z_{2n-1}+\overset{\vee}{\min_{c\le y\le d}} z_{2n-1}\right],$$

$$h_n=h_n(y)=\frac{1}{2}\left[\overset{\wedge}{\max_{a\le x\le b}} z_{2n}+\overset{\vee}{\min_{a\le x\le b}} z_{2n}\right],$$

$$\overset{\wedge}{\max} z=\sum_1^m\sum_1^n \max A_{pq}x^p y^q,$$

$$\overset{\vee}{\min} z=\sum_1^m\sum_1^n \min A_{pq}x^p y^q;$$

$\widehat{\max} z$ denotes the sum of the maxima of all terms of the polynomial, and $\check{\min} z$ the sum of the minima. Obviously,

$$\widehat{\max} z \geq \max z; \qquad \check{\min} z \leq \min z.$$

Denote

$$\lim_{n\to\infty} g_n(x) = g_0(x) = g_0, \qquad \lim_{n\to\infty} h_n(y) = h_0(y) = h_0.$$

We assert that

$$\|z - (g_0 - h_0)\| = E[z],$$

i.e. $g_0(x) + h_0(y)$ is a best approximating function.

We shall need some auxiliary facts.

Lemma 1. If, for nonnegative numbers c, d, y, for some fixed i,

$$y^i \geq \frac{1}{2}(c^i + d^i), \qquad y^{i+1} < \frac{1}{2}(c^{i+1} + d^{i+1}),$$

then

$$y^{i+k} < \frac{1}{2}(c^{i+k} + d^{i+k}), \qquad k = 2, 3, \ldots$$

Lemma 2. If, for nonnegative numbers c, d, and y,

$$y < \frac{1}{2}(c + d),$$

then

$$y^i < \frac{1}{2}(c^i + d^i), \qquad i = 2, 3, \ldots$$

Corollary. If $y^n \geq \frac{1}{2}(c^n + d^n)$, then

$$y^i \geq \frac{1}{2}(c^i + d^i), \qquad i = 1, 2, \ldots, n-1.$$

Indeed, the existence of at least one inequality

$$y^{i_0} < \frac{1}{2}(c^{i_0} + d^{i_0}), \qquad i_0 \leq n-1,$$

would imply, by one of Lemmas 1 and 2, that

$$y^n < \frac{1}{2}(c^n + d^n).$$

Theorem 2. Method α makes it possible to obtain, for the polynomial

$$z = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} x^p y^q, \qquad A_{pq} \geq 0,$$

the best approximating function of the form $\varphi(x) + \psi(y)$ on the rectangle $[a, b; c, d]$, $a, c \geq 0$, and this function is the polynomial

$$\frac{1}{2} \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \left[x^p (c^q + d^q) + y^q (a^p + b^p) - \frac{1}{2} (a^p + b^p)(c^q + d^q) \right].$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.