Reports of the Academy of Sciences of the USSR

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970. Volume 193, No. 3

UDC 517.946

MATHEMATICS

V. N. SHUKHMAN

THE CAUCHY PROBLEM FOR NONSTRICTLY HYPERBOLIC EQUATIONS

(Presented by Academician S. L. Sobolev on 12 I 1970)

In the main, our notation corresponds to ([1]). We consider the space R^{l+1} of vectors $x = (x_1, \dots, x_l)$; x_0 will sometimes be denoted by t. By S_t we denote the hyperplane $x_0 = t$. Further,

$$D_x^\beta = \partial^{|\beta|}/\partial x_0^{\beta_0} \cdots \partial x_l^{\beta_l}, \qquad |\beta| = \beta_0 + \dots + \beta_l.$$

Introduce the norms

$$|D^n f, S_t|^2 = c \sup_{|\beta| \le n} |D_x^\beta f_0, S_t|_2^2 = c \sup_{|\beta| \le n} \int_{S_t} |D_x^\beta f|^2 \, dx_1 \cdots dx_l,$$

$$\|D^n f, S_t\|^2 = c \sup_{|\beta| \le n,\ K_t \subset S_t} |D_x^\beta f, K_t|_2^2 = c \sup_{|\beta| \le n,\ K_t \subset S_t} \int_{K_t} |D_x^\beta f|^2 \, dx_1 \cdots dx_l,$$

where K_t is the unit cube in S_t.

Introduce the quasinorms

$$|D^{n,\infty} f, S_t, \rho| = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \sup_{|\sigma| = s} |D^n D_x^\sigma f, S_t|,$$

$$\|D^{n,\infty} f, S_t, \rho\| = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \sup_{|\sigma| = s} \|D^n D_x^\sigma f, S_t\|,$$

$$\sigma = (0, \sigma_1, \dots, \sigma_l).$$

In what follows we assume everywhere that $n > l/2$.

Let a formal series be given

$$\Phi(t,\rho)=\sum_{s=0}^{\infty}\frac{\rho^s}{s!}\Phi_s(t)$$

and let $\alpha>1$. Introduce the operator λ

$$\lambda\Phi(t,\rho)=\sum_{s=0}^{\infty}\frac{\rho^s}{(s!)^\alpha}\Phi_s(t).$$

We shall say that $\Phi(t,\rho)\in\Gamma^p(\alpha)$, p an integer ≥ 0, $\alpha>1$, if

$$\left(\frac{\partial}{\partial t}\right)^j\lambda\Phi(t,\rho)=\sum_{s=0}^{\infty}\frac{\rho^s}{(s!)^\alpha}\frac{\partial^j\Phi_s}{\partial t^j},\qquad j\leq p,$$

is a holomorphic function of ρ in some neighborhood of zero.

Let $f: X\to C$, where X is a strip in R^{l+1}, $0\leq x_0\leq T$, and C is the complex plane. We shall say that

$$f\in\Upsilon_2^{n(\alpha)}\ (\Upsilon_{[2]}^{n(\alpha)}),$$

if

$$|D^{n,\infty}f,S_t,\rho|\in\Gamma^0(\alpha)\quad\left(\|D^{n,\infty}f,S_t,\rho\|\in\Gamma^0(\alpha)\right).$$

Consider the Cauchy problem

$$P(x,D)u=f;\qquad D_0^ju\big|_{S_0}=0,\quad j<m; \tag{1}$$

m is the order of P.

Theorem 1. Suppose that $P(x,D)$ has coefficients of the Gevrey class $\gamma_{[2]}^{n(\alpha)}$ and, moreover,

$$P_m(x,\xi)=0 \tag{*}$$

has, with respect to ξ_0, the roots

$$\xi_0=\lambda_1(x,\xi'),\dots,\xi_0=\lambda_m(x,\xi');\qquad \xi'=(\xi_1,\dots,\xi_l),$$

where $P_m(x,D)$ is the principal part of the operator P.

We shall consider the case of a hyperbolic operator, i.e., all $\lambda_i(x,\xi')$ are real. If the $\lambda_i(x,\xi')$ satisfy the conditions:

1. $\lambda_i(x,\xi') \in C^\infty$ jointly in the variables (x,ξ') for $|\xi'| \neq 0$.
2. There exists
$$\lim_{\substack{x\to\infty \\ |\xi'|\neq 0}} \lambda_i(x,\xi') = \lambda_i(\infty,\xi')$$
and $\lambda_i(x,\xi') - \lambda_i(\infty,\xi') \in S$,

where S is the Shilov space.

3. $\lambda_i(x,\xi'/|\xi'|) \in \gamma_{[2]}^{n(\alpha)}(X)$ for all $|\xi'| \neq 0$, as functions of x.

Then the Cauchy problem (1) has a unique solution $u \in \gamma_2^{n(\alpha)}(Y)$ for every function $f \in \gamma_2^{n(\alpha)}(X)$; Y is the strip $0 \leq x_0 \leq T' \leq T$, for $1 \leq \alpha \leq m/(m-1)$.

The **proof** differs from the proof of the main theorem in (1) only in that the operator P is replaced by

$$P(x,D) \equiv \Lambda_1 \dots \Lambda_m + \Lambda,$$

where Λ_i are pseudodifferential operators with symbol

$$\xi_0 - \lambda_i(x,\xi'), \tag{2}$$

and Λ is a pseudodifferential operator of order $m-1$.

Preliminarily, for the operator with symbol (2), Gårding's estimate (2) is proved.

Theorem 2. Under the assumptions of Theorem 1, for $1 < \alpha \leq m/(m-1)$ there is a finite domain of dependence.

Remark 1. In the case of constant coefficients, Theorem 1 is known and is due to Gårding.

Theorem 3. Under the assumptions of Theorem 1, there exists a unique solution of the Cauchy problem (1),

$$u \in \gamma_2^{m+n-q(\alpha)}(Y),$$

for every function $f \in \gamma_2^{n(\alpha)}(X)$, in the strip Y in R^{l+1}:

$$0 \leq x_0 \leq T' \leq T;$$

q is the number of non-simple roots of equation $(*)$.

The **proof** is based on representing the operator P in the form

$$P = AB + C, \tag{3}$$

where A is strictly hyperbolic, B is a hyperbolic pseudodifferential operator, and C has arbitrarily small order. The decomposition is constructed as follows: from the roots $\xi_0 - \lambda_1(x,\xi'), \dots, \xi_0 - \lambda_m(x,\xi')$, choose $\xi_0 - \lambda_1, \dots, \xi_0 - \lambda_r$ such that

$$\lambda_i(x,\xi') \neq \lambda_j(x,\xi'), \quad i = 1,\dots,r; \quad j = 1,\dots,m, \quad |\xi'| \neq 0.$$

Form the functions

$$\nu(x,\xi) = \prod_{i=1}^{r} [\xi_0 - \lambda_i(x,\xi')]$$

and

$$\mu(x,\xi) = \prod_{i>r} [\xi_0 - \lambda_i(x,\xi')].$$

Obviously, $\nu^2 + \mu^2 \neq 0$ for $|\xi| \neq 0$. Assuming that the symbol of A is

$$a(x,\xi) = a_r(x,\xi) + \dots$$

and the symbol of B is

$$b(x,\xi) = b_{m-r}(x,\xi) + b_{m-r-1}(x,\xi) + \dots,$$

we set

$$a_r = \nu(x,\xi), \qquad b_{m-r} = \mu(x,\xi).$$

Using the formula for the symbol of the superposition of two operators (4), we see that a_{r-1} and b_{m-r-1} can be chosen as follows:

$$a_{r-1}(x,\xi) = \mu(x,\xi)Q(x,\xi)/\nu^r(x,\xi) + \mu^2(x,\xi),$$

$$b_{m-r-1}(x,\xi) = \nu(x,\xi)Q(x,\xi)/\mu^r(x,\xi) + \nu^2(x,\xi),$$

where

$$Q = P_{m-1}(x,\xi) + \sum_{i=0}^{l} \frac{\partial \nu(x,\xi)}{\partial x_i} \frac{\partial \mu(x,\xi)}{\partial \xi_i},$$

and, with this choice of $a_{r-1}(x,\xi)$, $b_{m-r-1}(x,\xi)$, the order of the operator $(P - AB)$ is at most $m-2$. Analogously one can choose also a_{r-2}, b_{m-r-2}, etc. It remains to apply induction on the order m of the equation under the assumption $q < m$ and to use Gårding' s inequality for pseudodifferential operators.

Theorem 4. Under the hypotheses of Theorem 1, there exists a unique solution of the Cauchy problem (1) $u \in \gamma_2^{n(\alpha)}$ for any function $f \in \gamma_r^{n(\alpha)}$ when $1 < \alpha \leq q/(q-1)$, where q is the maximal multiplicity of a root of the equation $P_m(x,\xi) = 0$.

Proof is based on a decomposition similar to decomposition (3):

$$P \equiv AB_1 \dots B_s + C,$$

where A is strictly hyperbolic, B_i are hyperbolic pseudodifferential operators, and the order of C is arbitrarily small.

Let us separate among the roots $\xi_0 - \lambda_{r+1}, \dots, \xi_0 - \lambda_m$ the series:

$$\xi_0 - \lambda_1^{(1)}, \dots, \xi_0 - \lambda_{r_1}^{(1)}; \quad \xi_0 - \lambda_1^{(r)}, \dots, \xi_0 - \lambda_{r_2}^{(r)}; \dots$$

such that

$$\lambda_i^{(s)}(x, \xi') \neq \lambda_j^{(k)}(x, \xi')$$

for $s \neq k, \quad i = 1, \dots, r_s, \quad j = 1, \dots, r_k, \quad x \in V, \quad \xi' \in R^l, \quad |\xi'| \neq 0;$
V is some neighborhood of the point x_0.

The operators B_i are constructed from these roots. After this, for the proof it remains to use a partition of unity in the space $\xi' = (\xi_1, \dots, \xi_l)$, induction on m under the assumption $q < m$, and so on.

Remark 2. The sharpness of the result of Theorem 4 for the whole class of hyperbolic equations whose characteristic polynomials have roots of multiplicity q can be proved analogously to the corresponding example in ([3]).

In conclusion, the author expresses his gratitude to V. Ivrii, V. Nalimov, and M. D. Ramazanov for numerous discussions.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
4 I 1970

CITED LITERATURE

1. I. Leray, Y. Ohya, Math. Ann., **170** (3), 67 (1967).
2. L. Gårding, *The Cauchy Problem for Hyperbolic Equations*, Moscow, 1961.
3. I. Leray, Math. Ann., **162** (2), 228 (1966).
4. J. J. Kohn, L. Nirenberg, in the collection *Pseudodifferential Operators*, Moscow, 1968, p. 9.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.