THE FUNDAMENTAL MATRIX OF AN ELLIPTIC SYSTEM OF SECOND ORDER WITH A COMPLEX PARAMETER

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.949.2

MATHEMATICS

M. L. RASULOV

THE FUNDAMENTAL MATRIX OF AN ELLIPTIC SYSTEM OF SECOND ORDER WITH A COMPLEX PARAMETER

(Presented by Academician I. N. Vekua on 12 XII 1969)

The fundamental matrix for a general elliptic system without a parameter was constructed in the work ([1]), whose method was also applied in the case of an elliptic system with a real parameter ([2]).

In connection with the application of the contour-integral method ([3]) to the solution of parabolic problems, the present note is devoted to the construction and estimation of the fundamental matrix for the system

$$A(x)\Delta u + \sum_{i=1}^{3} A_i(x)\frac{\partial u}{\partial x_i} + (A_0(x) - \lambda^2)u = 0, \tag{1}$$

considered in a three-dimensional domain D of Euclidean space, where $A(x), A_i(x)$ $(i = 0, 1, 2, 3)$ are square matrices of order m.

It is assumed that in the closed domain D the matrices $A(x), A_i(x)$ $(i = 0, 1, 2, 3)$ have continuous first-order derivatives with respect to all their arguments and that the roots ν_i $(i = 1, 2, \dots, p)$ of the characteristic equation

$$\Delta(1, \nu) = \det(A(x) + \nu E) = 0 \tag{2}$$

have constant multiplicity m_i and strictly negative real parts*

$$\operatorname{Re} \nu_i(x) < 0, \tag{3}$$

where $\Delta(\beta, \nu) = \det(\beta A(x) + \nu E)$, and E is the identity matrix of order m.

The fundamental matrix $\widetilde{P}_0(x - \xi, \xi, \lambda)$, with singularity at the point $x = \xi$, for the system

$$\sum_{i=1}^{3} A(\xi)\frac{\partial^2 u}{\partial x_i^2} - \lambda^2 u = 0$$

is constructed in finite form

$$P_0(x-\xi,\xi,\lambda) = \left(P_{0ks}(x-\xi,\xi,\lambda)\right)_{k,s=1}^{m}, \tag{4}$$

where the elements $P_{0ks}(x-\xi,\xi,\lambda)$ admit the representations

$$P_{0ks}(x-\xi,\xi,\lambda) = \frac{1}{4\pi|x-\xi|}\sum_{i=1}^{p}\sum_{j=1}^{m_i}\frac{B_{sk}^{(i,j)}(\xi)(m_i-j)!}{(-\nu_i(\xi))^{m_i-j+1}}$$
$$\times\left\{\frac{1}{(m_i-j)!}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-\nu_i(\xi)}}\right] + \sum_{r=1}^{m_i-j}\frac{1}{(r!)^2(m_i-j-r)!}\left[\left(-\frac{|x-\xi|}{2\sqrt{-\nu_i(\xi)}}\right)^r\lambda^r + \right.\right.$$

* The fundamental matrices for such an equation in the case of a constant matrix A of second and third orders were constructed in ([4,5]).

$$\left.\left. + \sum_{q=1}^{r}(-1)^q\left(\frac{|x-\xi|}{2\sqrt{-v_i(\xi)}}\right)^{r-q}\sum_{j_q=1}^{r-1}\sum_{j_{q-1}=q-1}^{j_q-1}\cdots\sum_{j_1=1}^{j_2-1}\left(\frac{j_{\nu+1}+\nu}{2}\right)\lambda^{r-q}\right]\times\right.$$
$$\times\exp\left(-\lambda\frac{|x-\xi|}{\sqrt{-v_i(\xi)}}\right); \tag{5}$$

$|x-\xi|$ is the length of the vector $x-\xi$,

$$B_{sk}^{(i,j)}(\xi) = \frac{1}{(j-1)!}\frac{\partial^{j-1}}{\partial v^{j-1}}\left.\frac{\Delta_{sk}(1,v)}{\prod\limits_{\substack{r=1\\ r\neq i}}^{p}(v-v_r(\xi))}\right|_{v=v_i(\xi)} \qquad (j=1,\ldots,m_i), \tag{6}$$

$\Delta_{ks}(\beta,\gamma)$ is the cofactor of the element (k,s) in the determinant $\Delta(\beta,\gamma)$.

Formula (5) can be written in a more transparent form:

$$P_{0ks}(x-\xi,\xi,\lambda) =$$

$$= -\frac{1}{4\pi|x-\xi|}\sum_{i=1}^{p}\sum_{j=1}^{m_i}\frac{B_{sk}^{(i,j)}(\xi)}{(m_i-j+1)!}\frac{\partial^{m_i-j}}{\partial v^{m_i-j}}\left.\frac{\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-v}}\right]}{v}\right|_{v=v_i(\xi)}. \tag{7}$$

In particular, if all roots $v_i(x)$ of the characteristic equation (2) are simple ($m_i = 1;\ i = 1, 2, \dots, m = p$), then from (7), taking (6) into account, we have

$$P_{0ks}(x-\xi,\xi,\lambda) = -\frac{1}{4\pi|x-\xi|}\sum_{i=1}^{m}\frac{\Delta_{sk}(1, v_i(\xi))}{\prod\limits_{\substack{r=1\\ r\neq i}}^{p}(v_i(\xi)-v_r(\xi))}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-v_i(\xi)}}\right]. \tag{8}$$

According to the condition on the roots $v_i(\xi)$ of the characteristic equation, the inequalities (3) hold, and, consequently, there exists a positive number δ such that for all $x, \xi \in \overline{D}$ the estimates

$$\left|\frac{\partial^k P_0(x-\xi,\xi,\lambda)}{\partial x_i^k}\right| \leq \frac{CB\exp(-\varepsilon|\lambda||x-\xi|)}{|x-\xi|^{k+1}} \qquad (k = 0, 1, 2); \tag{9}$$

$$\left|\frac{\partial P_0(x-\xi,\xi,\lambda)}{\partial \xi_i}\right| \leq \frac{CB\exp(-2\varepsilon|\lambda||x-\xi|)}{|x-\xi|^2}; \tag{10}$$

$$\frac{\partial^k}{\partial x_i^k}\left(\frac{\partial P_0(x-\xi,\xi,\lambda)}{\partial x_i} + \frac{\partial P_0(x-\xi,\xi,\lambda)}{\partial \xi_i}\right) \leq \frac{CB\exp(-\varepsilon|\lambda||x-\xi|)}{|x-\xi|^{1+k}} \qquad (k = 0, 1), \tag{11}$$

where λ is any value from the domain R_δ:

$$|\lambda| \geq R, \qquad |\arg\lambda| \leq \pi/4 + \delta; \tag{R_δ}$$

C, R are sufficiently large positive constants; B is a square matrix of order m, composed of ones; ε is some positive constant; inequalities (9)–(11) hold between the corresponding elements of the left- and right-hand sides.

The fundamental matrix $P(x,\xi,\lambda)$ of system (1), with a singularity at the point $x = \xi$, is constructed by the Levi–Carleman method ([6,7]) in the form

$$P(x,\xi,\lambda) = P_0(x-\xi,\xi,\lambda) + \int_D P_0(x-\eta,\eta,\lambda)\,h(\eta,\xi,\lambda)\,dD_\eta, \tag{12}$$

where $h(\eta,\xi,\lambda)$ is the unknown density of the integral correction

$$P_1(x,\xi,\lambda) = \int_D P_0(x-\eta,\eta,\lambda)\, h(\eta,\xi,\lambda)\, dD_\eta. \tag{13}$$

Substituting (12) into the left-hand side of equation (1) and (taking into account (9)–(11)) equating the resulting expression to zero, we arrive at the integral equation

$$h(x,\xi,\lambda) = K(x,\xi,\lambda) + \int_D K(x,\eta,\lambda) h(\eta,\xi,\lambda)\, dD_\eta, \tag{14}$$

where

$$K(x,\xi,\lambda) = \left\{ (A(x) - A(\xi))\Delta_x + \sum_{i=1}^{3} A_i(x) \frac{\partial}{\partial x_i} + A_0(x) \right\} P_0(x-\xi,\xi,\lambda).$$

According to (9), under the restrictions imposed, the following inequality holds for the kernel:

$$|K(x,\xi,\lambda)| \le CB \exp(-2\varepsilon|\lambda||x-\xi|)/|x-\xi|^2, \tag{15}$$

which is satisfied in the domain R_δ.

Estimate (15) makes it possible to construct a solution $h(x,\xi,\lambda)$ of the integral equation (14) in the domain R_δ by the method of successive approximations,

$$h(x,\xi,\lambda) = K(x,\xi,\lambda) + \sum_{n=2}^{\infty} K_n(x,\xi,\lambda), \tag{16}$$

where K_n are the iterations of the kernel K; moreover, for h the estimate

$$|h(x,\xi,\lambda)| \le CB \exp(-\varepsilon|\lambda||x-\xi|)/|x-\xi|^2 \tag{17}$$

holds in the domain R_δ.

Thus one proves the

Theorem. *Under the restrictions imposed, there exists a positive number δ such that, for sufficiently large R, for all complex values of λ satisfying the inequalities (R_δ), the system [1] has a fundamental matrix $P(x,\xi,\lambda)$ of the form (12), analytic in λ in the domain R_δ, where for $x,\xi \in \overline{D}$ the inequalities (9), (11) and the estimates*

$$|P_1(x,\xi,\lambda)| \le CB \exp(-\varepsilon|\lambda||x-\xi|)/\lambda|x-\xi|,$$

$$|\partial^k P_1(x,\xi,\lambda)/\partial x_i^k| \le CB\exp(-\varepsilon|\lambda||x-\xi|)/|x-\xi|^k \quad (k=1,2),$$

$$\left|\frac{\partial^k}{\partial x_i^k}\left[\frac{\partial P(x,\xi,\lambda)}{\partial x_i}+\frac{\partial P_0(x-\xi,\xi,\lambda)}{\partial \xi_i}\right]\right| \le \frac{CB\exp(-\varepsilon|\lambda||x-\xi|)}{|x-\xi|^{1+k}},$$

$$|\partial^k P(x,\xi,\lambda)/\partial x_i^k| \le CB\exp(-\varepsilon|\lambda||x-\xi|)/|x-\xi|^{1+k} \quad (k=0,1,2).$$

If $\Phi(x)$ is a vector-function having in D bounded continuous first-order derivatives with respect to all its arguments, then the vector-function

$$u(x,\lambda,\Phi) = -\int_D P(x,\xi,\lambda)\Phi(\xi)\,dD_\xi$$

for all λ belonging to R_δ is a solution of the nonhomogeneous equation

$$A(x)\Delta u + \sum_{i=1}^{3} A_i(x)\frac{\partial u}{\partial x_i} + (A_0(x)-\lambda^2)u = \Phi(x).$$

An analogous theorem also holds for the case of a domain D of an arbitrary number of dimensions greater than one, with the corresponding complications of the formulas and estimates given, and $P_{0ks}(x-\xi,\xi,\lambda)$ is expressed in terms of a Bessel function.

Azerbaijan State University
named after S. M. Kirov
Baku

Received
26 XI 1969

CITED LITERATURE

1. Ya. B. Lopatinskii, *Ukr. Mat. Zh.*, **3**, 3 (1951).
2. D. F. Melnik, *Nauk. Zap. LDU*, ser. mech.-math., **8** (1958).
3. M. L. Rasulov, *Method of the Contour Integral*, "Nauka," 1964.
4. M. L. Rasulov, *DAN*, **177**, No. 6 (1967).
5. M. L. Rasulov, *DAN*, **180**, No. 5 (1968).
6. E. E. Levi, *UMN*, vol. 8 (1941).

7. T. Carleman, *Ber. Sächs. Acad. Wiss. Leipzig, Math.-phys. Klasse*, **88** (1936).

Note: Figure translations are in progress. See original paper for figures.