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UNSTABLE EXTREMAL PROBLEMS

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Abstract

Full Text

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MATHEMATICS

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UNSTABLE EXTREMAL PROBLEMS AND GEOMETRIC PROPERTIES OF BANACH SPACES

(Presented by Academician A. N. Tikhonov on 21 IV 1970)

1. We consider the extremal problem

$$\inf_{x \in M} F(x) = F_0, \quad (1)$$

where x is an element of the real Banach space X , and $M \subseteq X$ is a convex closed set. Suppose that the functional $F(x)$ attains the minimal value F_0 on the set $\Omega_0 \subseteq M$, with Ω_0 nonempty. The optimization problem (1) is called stable (see ^(1,2)) if it is solvable and, for any sequence $\{x_n\}$ minimizing $F(x)$ on M , the condition

$$\rho(x_n, \Omega_0) = \inf_{x \in \Omega_0} \|x_n - x\| \rightarrow 0$$

is satisfied. It is known, however, that there are whole classes of extremal problems which, although solvable, belong to the class of unstable ones ⁽¹⁻⁵⁾.

The regularization method for constructing a strongly convergent minimizing sequence in such problems was first proposed by A. N. Tikhonov ^(1,2). A direct development of this method is given in the works ⁽³⁻⁵⁾. In the present note, regularization is carried out by constructing a sequence of sets, in each of which an element of least norm is sought. In terms of the geometry of the unit sphere of the B -space X , conditions are found that are necessary and sufficient for the existence and convergence of a minimizing sequence constructed in this way for a very broad class of functionals $F(x)$.

2. Everywhere below we use the following notation: U is the unit ball of the B -space X , and its boundary is

$$S = \{x \in X : \|x\| = 1\}.$$

The functional $F(x)$ is called **convex** if

$$F\left[\frac{x_1 + x_2}{2}\right] \leq \frac{1}{2}F(x_1) + \frac{1}{2}F(x_2)$$

for all x_1, x_2 ; $F(x)$ is called a **quasiconvex** functional if, for any x_1, x_2 ,

$$F\left[\frac{x_1 + x_2}{2}\right] \leq \max\{F(x_1), F(x_2)\}.$$

Clearly, a convex functional is quasiconvex, whereas the converse is, in general, false. One says that the B -space X is **strictly convex** if S contains no line segments (of course, not degenerating to a point).

Following ⁽⁶⁾, we shall call a set $K \subseteq X$ **approximately compact** if, for every $x \in X$, any sequence $\{y_n\} \subseteq K$ such that $\rho(x, y_n) \rightarrow \rho(x, K)$ has a limit point $y \in K$. Following ⁽⁷⁾, we shall say that the B -space X has the Efimov-Stechkin property if every sequentially weakly closed set in X is approximately compact. A Banach space X is called an E -space if it is strictly convex and has the Efimov-Stechkin property ⁽⁷⁾.

It is shown in ⁽⁶⁾ that uniformly convex B -spaces are E -spaces, for example L_p , $p > 1$. On the other hand, examples are known of strictly convex reflexive spaces that do not possess the Efimov-Stechkin property and are therefore not E -spaces.

properties ^(7, 8). Note that E -spaces are the largest known class of Banach spaces for which the operator P of metric projection onto a closed convex set is correct in the sense of Hadamard (see ⁽⁹⁾). Let us give one equivalent definition of an E -space ⁽¹⁰⁾: a space X is an E -space if it is reflexive, strictly convex, and from the conditions $\{x_n\} \subset X$, $x \in X$, $\|x_n\| = \|x\| = 1$, and $x_n \rightarrow x$ (weakly) it follows that $\|x_n - x\| \rightarrow 0$.

3. We now consider a regularization algorithm for problem (1). We shall assume that there is a method allowing one to solve the optimization problem (1) with arbitrarily high accuracy in the functional, i.e., that for any $\delta > 0$ an R_δ is given, defined by the relation $|F_0 - F_\delta| \leq \delta$. To problem (1) we assign the following problem: find

$$\inf \|x\| \quad \text{under the condition } x \in \Omega_\delta,$$

where

$$\Omega_\delta = \{x \in M : F(x) \leq F_\delta + \delta\}. \quad (2)$$

Theorem 1. *In a Banach space X the following assertions are equivalent:*

- a) *problem (2) has a unique solution x_δ for any $\delta > 0$, F_δ , convex closed set M , and quasiconvex lower semicontinuous functional $F(x)$;*
- b) *the space X is strictly convex and reflexive.*

Theorem 2. *In a strictly convex reflexive B -space X the following assertions are equivalent:*

- a) the sequence $\{x_\delta\}$ of solutions of problem (2) converges strongly as $\delta \rightarrow 0$ to x_0 , the element of the set Ω_0 with least norm, for any F_δ , convex closed set M , and quasiconvex lower semicontinuous functional $F(x)$;
- b) the space X is an E -space.

We shall call problem (2) stable if, for any fixed $\delta > 0$, F_δ , it has a unique solution x_δ , and every sequence $\{x_n\} \subset \Omega_\delta$ such that $\|x_n\| \rightarrow \rho(0, \Omega_\delta)$ converges strongly to x_δ . This stability property is very important, since it makes it possible to solve problem (2) of finding x_δ approximately by means of any method minimizing $\|x\|$ on the set Ω_δ (one of such methods we consider in Sec. 4).

Theorem 3. *In a strictly convex B -space X the following assertions are equivalent:*

- a) problem (2) is stable, whatever the convex closed set M and quasiconvex lower semicontinuous functional $F(x)$ may be;
- b) the space X is an E -space.

By definition, we shall call a set $R \subseteq M$ a class of stabilization for the extremal problem (1) if, for every sequence $\{x_n\} \subset R$ minimizing $F(x)$ on M , the condition $\rho(x_n, \Omega_0) \rightarrow 0$ is satisfied. From the results of A. N. Tikhonov's work (2) it follows that a stabilization class is, for example, any compact $R \subseteq M$ containing a unique point from Ω_0 . Besides topological ones, in Banach spaces one can indicate geometric characteristics of stabilization classes for extremal problems. We formulate, in terms of the geometry of the unit sphere of the B -space X , conditions necessary and sufficient for stabilization classes to be the sets $R_0 = \rho_0 U \cap M$, where $\rho_0 = \rho(0, \Omega_0)$.

Theorem 4. *In a strictly convex B -space X the following assertions are equivalent:*

- a) the set R_0 is nonempty and is a stabilization class for the extremal problem (1), whatever the convex closed set M and quasiconvex lower semicontinuous functional $F(x)$ may be;
 - b) the space X is an E -space.
4. One of the methods for the numerical determination of elements x_δ solving problem (2) is passage to a certain finite-dimensional analogue and the finding of approximations to x_δ that are elements of finite-dimensional sets. Suppose that there is an increasing chain of convex closed finite-dimensional sets

$$M_n \subset M_{n+1} \subset \dots \subset M, \quad n = 1, 2, \dots,$$

such that

$$\overline{\bigcup_{n=1}^{\infty} M_n} = M.$$

In computing the functional $F(x)$, one usually uses its approximate values, i.e., finds $F^{(k)}(x)$, such that

$$F^{(k)}(x) \rightarrow F(x) \quad (k \rightarrow \infty). \quad (3)$$

We associate with problem (2) the following finite-dimensional variational problem: find

$$\inf \|x\| \quad \text{under the condition } x \in \Omega_{\delta,n}^k,$$

where

$$\Omega_{\delta,n}^k = \{x \in M_n : F^{(k)}(x) \leq F_{\delta} + \delta\}. \quad (4)$$

Theorem 5. Suppose X is an E -space, the functional $F(x)$ is convex and continuous on M , the functionals $F^{(k)}(x)$ are lower semicontinuous on the sets M_n , and the convergence (3) takes place uniformly, for fixed n , on every bounded subset of M_n ($n = 1, 2, \dots$). Then there exist n_0 and k_0 such that for $n > n_0$ and $k > k_0$ problem (4) has a solution $x_{\delta,n}^k$. Moreover,

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} x_{\delta,n}^k = x_{\delta}.$$

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Note: Figure translations are in progress. See original paper for figures.

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