ON THE STABILITY OF SELF-OSCILLATIONS OF A FLUID

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Abstract

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ON THE STABILITY OF SELF-OSCILLATIONS OF A FLUID

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A linearization law is established in the stability problem for a periodic self-oscillation of a viscous incompressible fluid. An infinite-dimensional analogue of the Andronov—Witt theorem (^{1,2}) is proved.

1. The general stability problem for a cycle. Let a dynamical system (Q_t, X) be given, i.e., a nonlinear partial semigroup of operators $Q_t : X \to X$ $(0 \le t < \infty)$, acting in a Banach space X. This means that for each $x \in X$ an element $Q_t x$ is defined for $0 \le t < t_0(x)$, with $Q_0 x = x$; $Q_{t+\tau}x = Q_{tQ_\tau}x$ $(t, \tau \ge 0;\ t + \tau < t_0(x))$.

Suppose that the operator Q_T for some $T > 0$ has a fixed point q_0. Then $q_\tau = Q_\tau q_0$ is also a fixed point of the operator Q_T for all $\tau > 0$. If $C = \bigcup_{0 \le \tau \le T} q_\tau$ contains more than one point, we shall say that C is a cycle (or self-oscillation) of the dynamical system (Q_t, X). We shall call it (asymptotically) stable if, for every $\varepsilon > 0$, one can indicate such a $\delta > 0$ that from $\rho(x_0, C) < \delta$ it follows that $t_0(x_0) = \infty$ and $\rho(Q_t x_0, C) < \varepsilon$ for $t \ge 0$ (and $\rho(Q_t x_0, C) \to 0$ as $t \to +\infty$). If $\|Q_t x_0 - q_{t+h}\| \to 0$ as $t \to +\infty$, then we shall say that the trajectory $Q_t x_0$ has asymptotic phase h.

Next, we shall call the cycle C smooth if in some neighborhood of it $\omega_\eta = \{x \in X : \rho(x, C) < \eta\}$ the operators Q_t are continuously Fréchet differentiable (with respect to x), the derivative $d/d\tau\, q_\tau = \dot{q}_\tau$ exists $(0 \le \tau \le T)$, and, moreover, for any $\varepsilon_1, \varepsilon_2 > 0$ there exist $\delta_1, \delta_2 > 0$ such that from $\|a\| < \delta_1$, $|s| < \delta_2$ follow estimates, uniform in $\tau \in [0, T]$,

$$\|\Delta_1(\tau, a)\| \equiv \|Q_T(q_\tau + a) - q_\tau - Q'_T(q_\tau)a\| \le \varepsilon_1 \|a\|; \tag{1}$$

$$\|\Delta_2(\tau, s)\| \equiv \|q_{\tau+s} - q_\tau - \dot{q}_\tau s\| \le \varepsilon_2 |s|. \tag{2}$$

The operator $U_{T,\tau} = Q'_T(q_\tau)$ will be called the monodromy operator (corresponding to the initial instant τ). Differentiating the equality $Q_{Tq_\tau} = q_\tau$ with respect to τ, we infer that $\varphi_\tau = \dot{q}_\tau$ is an eigenvector of the operator $U_{T,\tau}$ corresponding to the eigenvalue 1: $U_{T,\tau}\varphi_\tau = \varphi_\tau$.

Lemma 1. *The spectrum of the monodromy operator $U_{T,\tau}$ does not depend on τ.*

This lemma is easily derived by taking into account the equalities

$$U_{T,\tau} = Q'_\tau(q_0)Q'_{T-\tau}(q_\tau), \qquad U_{T,0} = Q'_{T-\tau}(q_\tau)Q'_\tau(q_0) \tag{3}$$

and using the following simple assertion.

Lemma 2. *Let $U, V : X \to X$ be linear bounded operators. Then the operators UV and VU have one and the same nonzero spectrum: $\sigma(UV) - \{0\} = \sigma(VU) - \{0\}$. Moreover, if 0 is excluded, the point, continuous, and residual spectra, respectively, coincide. The multiplicities of the nonzero eigenvalues are then the same.*

Theorem 1. *Let $C = \bigcup_{0\le\tau\le T} Q_\tau q_0$ be a smooth cycle of the dynamical system (Q_t, X), and let the spectrum of the monodromy operator $U_{T,\tau}$ have the form*

$$\sigma(U_{T,\tau}) = \{1\} \cup \sigma_0(U_{T,\tau}), \qquad |\sigma_0(U_{T,\tau})| < \alpha < 1. \tag{4}$$

Let 1 be a simple proper number. Then the cycle C is asymptotically stable, and every trajectory $\{Q_t x^0\}$, $t \ge 0$, has an asymptotic phase, provided only that the quantity $\rho(x_0, C)$ is sufficiently small.

Proof. Let ψ_τ be a fixed vector of the adjoint operator $U^*_{T,\tau}$, normalized by the condition $(\varphi_\tau, \psi_\tau) = 1$. Define the operator V_τ by setting

$$V_\tau x = U_{T,\tau}x - (x, \psi_\tau)\varphi_\tau. \tag{5}$$

It is clear that $\sigma(V_\tau) = \sigma_0(U_{T,\tau})$. Therefore, for sufficiently large natural m, the operator V_τ^m is a contraction (uniformly in τ): $\|V_\tau^m\| < \theta < 1$. We shall assume that this already holds for $m = 1$; this case can be attained by introducing the substitution $T \to mT$.

Let $\rho(x_0, C) < \delta$. Then for some τ_0, $0 \le \tau_0 \le T$, we have

$$\|x_0 - q_{\tau_0}\| < \delta. \tag{6}$$

Define sequences of time instants τ_n and elements a_n of the space X by setting

$$\tau_{n+1} = \tau_n + s_n, \qquad s_n = (a_n, \psi_{\tau_n}), \qquad a_n = x_n - q_{\tau_n}, \quad n = 0, 1, ... \tag{7}$$

We shall show that, if δ is sufficiently small, then the estimates

$$\rho(Q_T^n x_0, C) \le \|a_n\| \le \theta^n \delta, \qquad |s_n| \le l\theta^n \delta, \qquad l = \max \|\psi_\tau\| \tag{8}$$

hold.

For $n = 0$, the estimates (8) follow immediately from (6). If the estimates (8) have already been proved for $n = k$, then for $n = k + 1$ we derive them, using conditions (1) and (2), from the relation

$$\rho(x_{k+1}, C) \le \|a_{k+1}\| = \|V_{\tau_k} a_k + \Delta_1(\tau_k, a_k) + \Delta_2(\tau_k, s_k)\|.$$

Here it is sufficient to choose $\varepsilon_1, \varepsilon_2, \delta$ so small that the inequalities

$$\max_\tau \|V_\tau\| + \varepsilon_1 + l\varepsilon_2 < \theta, \qquad \delta < \delta_1, \qquad l\delta < \delta_2 \tag{9}$$

are satisfied.

It is now not difficult to establish that

$$\|Q_t x_0 - q_{h+t}\| \to 0 \quad (t \to +\infty), \qquad h = \tau_0 + \sum_{n=0}^{\infty} s_n, \tag{10}$$

which completes the proof.

Theorem 2. *Let C be a smooth cycle of the dynamical system (Q_t, X). Let the mapping Q_t be differentiable with respect to t, and let the derivative $\frac{d}{dt}Q_t = \dot{Q}_t$ be continuous in (x, t): $x \in \omega_n$, $0 \le t < t_0(x)$. Let the spectrum of the monodromy operator have the form*

$$\sigma(U_{T,\tau}) = \sigma_1(U_{T,\tau}) \cup \sigma_2(U_{T,\tau}); \quad |\sigma_1(U_{T,\tau})| > \beta > 1, \quad |\sigma_2(U_{T,\tau})| \le 1. \tag{11}$$

Then the cycle C is unstable.

Proof. Let $\psi \in X^*$ and $(\varphi_0, \psi) = 1$ (it is easy to prove that $\varphi_0 = \dot{q}_0 \ne 0$). Consider in the space X the plane $X_0 = \{a \in X : (a, \psi) = 0\}$ and the hyperplane $\Gamma = \{x : x = q_0 + a;\ a \in X_0\}$. Define a mapping K of a neighborhood of zero in the space X_0 into X_0 by setting

$$Ka = Q_{t_*}(q_0 + a) - q_0. \tag{12}$$

Here $t_* = t_*(x)$ is the instant of the first return of the trajectory $Q_t x$, $x \in \Gamma$, to the hyperplane Γ. To prove the existence of t_*, it is sufficient to apply the implicit-function theorem to the equation

$$F(t,a) \equiv (Q_t(q_0+a)-q_0,\psi) = 0. \tag{13}$$

Indeed, $F(T,0)=0$, $F_t(T,0)=(\dot{q}_0,\psi)=1$, and the function F is continuously differentiable in a neighborhood of the point $(T,0)\in R\times X_0$.

The operator K is continuously differentiable in a neighborhood of zero, and

$$K'(0)a = U_{\tau,0}a - \varphi_0(U_{\tau,0}a,\psi), \qquad a\in X_0. \tag{14}$$

The spectrum of the operator $K'(0)$ obviously contains the set $\sigma_1(U_{\tau,\tau})$. Now Theorem 2 is easily derived from Lemma 5 of paper $(^3)$, if one further notes that for points $x=q_0+a$, $a\in X_0$; $\|a\|<\varepsilon$, for sufficiently small $\varepsilon>0$, there exists a constant $\gamma>0$ such that $\rho(x,C)\geq\gamma\|a\|$.

2. Application to the Navier–Stokes equations. Let a viscous incompressible homogeneous fluid fill a three-dimensional bounded domain Ω with boundary S of class C^2. Let the body forces and the boundary value of the velocity be prescribed and independent of time. Then the Navier–Stokes equations and boundary conditions have the form

$$v_t + (v,\nabla)v - \nu\Delta v = -\nabla P + F(x), \qquad x\in\Omega; \tag{15}$$

$$\operatorname{div} v = 0; \tag{16}$$

$$v|_S = a(x). \tag{17}$$

Suppose that there exists a (sufficiently smooth) T-periodic in time t self-oscillatory solution of system (15)–(17), with velocity vector $v_0(x,t)$ and pressure $P_0(x,t)$. We shall be interested in its stability with respect to perturbations from the Hilbert space H_1. The latter is the closure of the set of smooth solenoidal vector fields vanishing on the boundary in the metric

$$(u,v)_{H_1} = \int_\Omega \sum_{k=1}^{3} \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}\,dx. \tag{18}$$

We shall call the cycle $C=\bigcup_t v_0(\cdot,t)$ (asymptotically) stable in H_1, if for every $\varepsilon>0$ there corresponds a $\delta>0$ such that

$$\rho(v(\cdot,t),C) = \inf_\tau \|v(\cdot,t)-v_0(\cdot,\tau)\|_{H_1} < \varepsilon$$

for any solution of system (15)–(17) with velocity vector v, provided that $\rho(v(\cdot,0),C)<\delta$.

The stability spectrum $\Sigma(v_0)$ (^{3}) is the set of those σ for which there exists a nonzero T-periodic solution of the linearized system

$$u_t + \sigma u + (v_0, \nabla)u + (u, \nabla)v_0 - \nu\Delta u = -\nabla q, \qquad \operatorname{div} u = 0, \qquad u|_S = 0. \quad (19)$$

Obviously, $\sigma_k = -2k\pi i/T \in \Sigma(v_0)$ $(k = 0, \mp 1, \ldots)$: the corresponding solution of system (19) is $v_{0t} \exp \sigma_k t,\ P_{0t} \exp \sigma_k t$. Applying Theorems 1 and 2, we arrive at the following conclusions.

Theorem 3. *Let $\sigma_0 = 0$ be a simple eigenvalue of system (19), and let all points of the stability spectrum $\Sigma(v_0)$, except $\sigma_k: \ k = 0, \mp 1, \ldots$, lie inside the left half-plane. Then the cycle C is asymptotically stable in H_1.*

Theorem 4. *If the stability spectrum $\Sigma(v_0)$ contains at least one point σ with $\operatorname{Re} \sigma > 0$, then the cycle C is unstable.*

For examples of self-oscillatory regimes, see ($^{4-7}$).

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Note: Figure translations are in progress. See original paper for figures.