ON AN INVERSE PROBLEM OF SPECTRAL ANALYSIS FOR THE HILL EQUATION

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Abstract

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ON AN INVERSE PROBLEM OF SPECTRAL ANALYSIS FOR THE HILL EQUATION

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1°. The Hill operator is the minimal closed differential operator H, generated in the Hilbert space $L_2(-\infty, \infty)$ by the differential expression

$$l[y] = -y''(x) + q(x)y(x), \tag{1}$$

where $q(x)$ is a periodic function of period a.

We shall assume that $q(x)$ is a real-valued function, bounded on the interval $[0, a]$. Then the operator H is self-adjoint and its spectrum is well studied (see, for example, (^{1})).

Denote by $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ the solutions of the Hill equation

$$l[y] = \lambda y, \tag{2}$$

satisfying at $x = 0$ the boundary conditions $\theta'(0, \lambda) = \varphi(0, \lambda) = 0$; $\theta(0, \lambda) = \varphi'(0, \lambda) = 1$, and by θ, θ', φ, and φ' the values which these functions, together with their derivatives with respect to x, take at the point $x = a$. The parameter λ is omitted for simplicity of notation. The function

$$\Delta(\lambda) = \theta + \varphi' \tag{3}$$

is called the discriminant of the Hill equation. Put

$$\Phi_\pm(\lambda) = \Delta(\lambda) \mp 2. \tag{4}$$

Let λ_j $(j = 0, 1, 2, ...)$ be the zeros of the function $\Phi_+(\lambda)$, arranged in nondecreasing order of their real parts, and let μ_j be the zeros of the function $\Phi_-(\lambda)$, ordered in the same way. It is known (^{1}) that λ_j and μ_j are real numbers satisfying the inequalities

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_3 \le \mu_4 < \cdots. \tag{5}$$

The intervals

$$Z_{2k} = [\lambda_{2k}, \mu_{2k}], \qquad Z_{2k+1} = [\mu_{2k+1}, \lambda_{2k+1}] \quad (k = 0, 1, ...) \tag{6}$$

are called intervals of stability. The intervals

$$\widetilde{Z}_{2k} = (\lambda_{2k-1}, \lambda_{2k}), \qquad \widetilde{Z}_{2k+1} = (\mu_{2k}, \mu_{2k+1}) \quad (k = 0, 1, ...;\ \lambda_{-1} = -\infty) \tag{7}$$

are called intervals of instability. The set of intervals of stability forms the spectrum of the operator H.

A number of works have been devoted to the question of determining the Hill equation from its discriminant. In the special case when all the intervals of instability disappear, it was proved in ([5,6]) that $q(x)$ is a constant.

In the case when only a finite number of intervals of instability do not disappear, the properties of the potential $q(x)$ were studied in ([5]). M. G. Krein ([2]), studying the inverse problem of spectral analysis for the equation of oscilla-

of an infinite string with a periodic distribution of masses, showed that even in the case of a symmetric string there exists an infinite set of strings with one and the same discriminant. Taking into account the connection between the equation of oscillations of a string and an equation of Sturm-Liouville type, it follows from the results of M. G. Krein that the discriminant $\Delta(\lambda)$ cannot in general determine the potential $q(x)$ uniquely.

A natural question arises: what spectral characteristics, besides the discriminant $\Delta(\lambda)$, must be known in order that the potential $q(x)$ be determined uniquely by them?

In the present paper one of the solutions of this problem is given.

2°. We define the second-order matrix $\sigma(\lambda) = \|\sigma_{ij}(\lambda)\|_{i,j}^{1,2}$ with elements $\sigma_{ij}(\lambda)$ that are analytic functions of the complex variable λ:

$$\sigma(\lambda) = \begin{pmatrix} \varphi & \frac{1}{2}(\varphi' - \theta) \\ \frac{1}{2}(\varphi' - \theta) & -\theta' \end{pmatrix}. \tag{8}$$

Using the equality $\theta\varphi' - \varphi\theta' = 1$, it is not hard to show that

$$\det \sigma(\lambda) = [4 - \Delta(\lambda)^2]/4. \tag{9}$$

The matrix $\sigma(\lambda)$, defined by formula (8), will henceforth be called the σ-matrix. This matrix plays an important role in the spectral theory of Hill' s equation, since with its help the spectral matrix $\rho(\lambda)$ (1) for the operator H is determined:

$$\rho(\lambda)=\begin{cases}(-1)^k 4\sigma(\lambda)(\det\sigma(\lambda))^{-1/2}, & \lambda\in Z_k \quad (k=0,1,...),\\ 0, & \lambda\in\widetilde{Z}_j \quad (j=0,1,...).\end{cases} \tag{10}$$

In the present article we shall solve the problem of reconstructing the periodic potential $q(x)$ from certain elements of the σ-matrix.

Theorem 1. *Let $q_j(x)\in L_1(0,a)$, $j=1,2$, be real-valued functions, and let H_1 and H_2 be Hill operators with potentials $q_1(x)$ and $q_2(x)$. Denote by $\sigma_j(\lambda)$ the σ-matrices for these operators. Then, if $\sigma_1(\lambda)=\sigma_2(\lambda)$, it follows that $q_1(x)=q_2(x)$ almost everywhere.*

Proof. By the hypotheses of Theorem 1, $\sigma_1(\lambda)=\sigma_2(\lambda)$. Introduce the notation

$$\sigma(\lambda)=\sigma_1(\lambda)=\sigma_2(\lambda)=\begin{pmatrix}\sigma_{11}(\lambda) & \sigma_{12}(\lambda)\\ \sigma_{12}(\lambda) & \sigma_{22}(\lambda)\end{pmatrix}.$$

Since $\sigma(\lambda)$ is the σ-matrix for the Hill operator, it must be that $\sigma_{11}(\lambda)=\varphi(\lambda)$; $\sigma_{12}(\lambda)=\frac{1}{2}(\varphi_p'-\theta_p)$; $\sigma_{22}(\lambda)=-\theta'$; $\det\sigma(\lambda)=[4-(\varphi_p+\theta_p)^2]/4$, where $\varphi(\lambda)\equiv\varphi_p$, φ_p', θ_p, $\theta'\equiv\theta_p'$ are the values at the point $x=a$ of the solutions $\theta_p(x,\lambda)$ and $\varphi_p(x,\lambda)$, constructed for the equations $-y''+q_p(x)y=\lambda y$.

Let γ be a zero of the function $\sigma_{11}(\lambda)$. This zero belongs to one of the intervals of instability $\widetilde{Z}_j$, and $\operatorname{sign}(\varphi_p'+\theta_p)(\gamma)=(-1)^j$. Taking this equality into account, we find

$$\varphi_p'(\gamma)-\theta_p(\gamma)=2\sigma_{12}(\gamma); \qquad \varphi_p'(\gamma)+\theta_p(\gamma)=(4-4\det\sigma(\gamma))^{1/2}(-1)^j, \tag{*}$$

where $\sigma_{12}(\gamma)$ and $\det\sigma(\gamma)$ are regarded as known quantities. Solving the system of equations $(*)$ with respect to the quantities $\varphi_p'(\gamma)$ and $\theta_p(\gamma)$, we find the values of the functions φ_p' and θ_p at the points γ, where $\varphi(\gamma)=0$. As a result we obtain that to two different potentials $q_1(x)$ and $q_2(x)$ there corresponds one and the same function $\varphi(\lambda)=\sigma_{11}(\lambda)$, and the values of the function φ_1' and φ_2' coincide: $\varphi_1'(\gamma)=\varphi_2'(\gamma)$ at all points γ, where $\varphi(\gamma)=0$.

Thus, we have arrived at two boundary-value problems:

$$-y''+q_p(x)y=\lambda y, \qquad y(0)=y(a)=0 \quad (p=0,1),$$

which have one and the same spectrum, determined by the zeros γ_n of the function $\varphi(\lambda)$, and one and the same spectral function $\tau(\lambda)$, whose jumps a_n at the points γ_n are determined by the numbers

$$a_n = \left.\frac{d\varphi(\lambda)}{d\lambda}\right|_{\lambda=\gamma_n} \varphi'(\gamma_n),$$

where $\varphi'(\gamma_n) = \varphi_p'(\gamma_n)$.

By virtue of the uniqueness theorem (see, for example, (4)), we find $q_1(x) = q_2(x)$ for $0 \leq x \leq a$ almost everywhere, and by virtue of periodicity $q_1(x) = q_2(x)$ for all $-\infty < x < \infty$. The theorem is proved.

Theorem 2. *In order to determine completely the σ-matrix of the Hill operator, it is sufficient to prescribe the following elements of it: 1) the diagonal element $\sigma_{11}(\lambda)$ (or $\sigma_{22}(\lambda)$); 2) $\det\sigma(\lambda)$; 3) the signs of the off-diagonal element $\sigma_{12}(\lambda)$ at those points λ where $\sigma_{11}(\lambda) = 0,\ \det\sigma(\lambda) \neq 0$.*

Proof. Let $\sigma_{11}(\gamma) = \varphi(\gamma) = 0$. Then $\theta\varphi' = 1$. Hence we find $4\sigma_{12}(\gamma)^2 = (\theta - \varphi')^2 = -4\det\sigma(\gamma) > 0$. Therefore the system of equations (*) can be rewritten in the form

$$\begin{aligned} \varphi'(\gamma) - \theta(\gamma) &= 2|\det\sigma(\gamma)|^{1/2}\operatorname{sign}(\varphi' - \theta), \\ \varphi'(\gamma) + \theta(\gamma) &= (4 - 4\det\sigma(\gamma))^{1/2}(-1)^j. \end{aligned}$$

Consequently, the values of the function φ' at the points where $\varphi = 0$ are determined by specifying $\det\sigma(\gamma)$ and the function $\operatorname{sign}(\varphi' - \theta)$.

Repeating verbatim the end of the proof of Theorem 1, we arrive at Theorem 2.

Corollary 1. *The discriminant $\Delta(\lambda)$ determines the Hill equation uniquely only in the case when all intervals of instability disappear.*

Theorem 3. *Let two entire functions $D(\lambda)$ and $\varphi(\lambda)$ be given, and let a function $\chi(\gamma)$ be defined at the points γ at which $\varphi(\gamma) = 0$, taking the value $+1$ or -1 for those γ for which $D(\gamma) \neq 0$, and the value zero if $D(\gamma) = 0$.*

In order that there exist a Hill operator with potential $q(x)$ of period a, where $q^{(m)}(x) \in L_1(0, a)$, $m \geq 1$, and with σ-matrix $\sigma(\lambda)$ such that $D(\lambda) = \det\sigma(\lambda)$, $\varphi(\lambda) = \sigma_{11}(\lambda)$ and $\operatorname{sign}\sigma_{12}(\gamma) = \chi(\gamma)$, it is necessary and sufficient that the following conditions be fulfilled:

1) *The function $\varphi(\lambda)$ is representable in the form*

$$\varphi(\lambda) = a\prod_{k=0}^{\infty}\frac{\gamma_k - \lambda}{(k+1)^2c^2}, \tag{11}$$

where γ_k are real numbers, $\gamma_k < \gamma_j,\ k < j$, satisfying the asymptotic formulas

$$\gamma_k = (k+1)^2 c^2 + a_0 + o(1), \tag{12}$$

$c = \pi/a,\ a_0$ is a constant number.

2) *The function* $D(\lambda)$ *is representable in the form*

$$D(\lambda) = \Phi_+(\lambda)\Phi_-(\lambda), \tag{13}$$

where

$$\Phi_+(\lambda) = 4\prod_{k=0}^{\infty} \frac{\mu_{2k} - \lambda}{(2k+1)^2 c^2} \prod_{k=0}^{\infty} \frac{\mu_{2k+1} - \lambda}{(2k+1)^2 c^2}; \tag{14}$$

$$\Phi_-(\lambda) = a^2 \lambda_0 \left(1 - \frac{\lambda}{\lambda_0}\right) \prod_{k=1}^{\infty} \frac{\lambda_{2k} - \lambda}{4k^2 c^2} \prod_{k=0}^{\infty} \frac{\lambda_{2k-1} - \lambda}{4k^2 c^2}; \tag{15}$$

here μ_ν, λ_ν *are real numbers such that*

$$\lambda_{2n-1(2n)} = (2n)^2 c^2 + a_0 + o(1), \tag{16}$$

$$\mu_{2n(2n+1)} = (2n+1)^2 c^2 + a_0 + o(1). \tag{17}$$

3) *The equalities hold*

$$\Phi_+(\lambda_\nu) = -4, \qquad \Phi_-(\lambda_\nu) = 4 \quad (\nu = 0, 1, 2, \ldots). \tag{18}$$

4) *The numbers* $\gamma_n, \mu_n, \lambda_n$ *satisfy the inequalities*

$$\lambda_0 < \mu_0 \le \gamma_0 \le \mu_1 < \lambda_1 \le \gamma_1 \le \lambda_2 < \mu_2 \le \gamma_2 \le \mu_3 < \lambda_3 < \cdots.$$

* It can be shown that at points λ where $\sigma_{11}(\lambda) = 0,\ \det\sigma(\lambda) = 0$, one has $\sigma_{12}(\lambda) = 0$.

5) The function

$$F(x,y) = \sum_{n=0}^{\infty} \left(\frac{\sin\sqrt{\gamma_n}\, x \cdot \sin\sqrt{\gamma_n}\, y}{\gamma_n B_n} - \frac{2}{\pi}\sin(n+1)x \sin(n+1)y \right), \tag{19}$$

where

$$B_n = (-1)^n \left. \frac{d\varphi(\lambda)}{d\lambda} \right|_{\lambda=\gamma_n} \left((-1)^n \chi(\gamma_n) D^{1/2}(\gamma_n) + \sqrt{-D(\gamma_n)+1}\right), \qquad (20)$$

has $(m+1)$-summable derivatives in the domain $(0 \leq x, t \leq \pi)$.

Theorem 4. If

$$\sqrt{\gamma_n} = (n+1)c + A_0/n + A_1/n^3 + O(1/n^4), \qquad (21)$$

$$\lambda_{2n-1(2n)} = (2n)^2 c^2 + a_0 + O(1/n^{1+\delta}), \qquad (22)$$

$$\mu_{2n+1(2n)} = (2n+1)^2 c^2 + a_0 + O(1/n^{1+\delta}), \qquad \delta > 0, \qquad (23)$$

then there exists a Hill operator with an absolutely continuous function $q(x)$.

3°. The solution of the inverse problem for the Hill equation on the whole line is essentially connected with the inverse problem for the Hill equation on the half-line.

We shall consider in the space $L_2(0,\infty)$ the self-adjoint operator H_+ generated by the differential expression (1) and the boundary condition $y(0) = 0$.

Theorem 5. Suppose that three sequences of real numbers $\lambda_\nu, \mu_\nu, \gamma_\nu$ are given, satisfying conditions (11)–(18) and condition 5) of Theorem 3, and suppose that the strict inequalities

$$\lambda_0 < \mu_0 < \gamma_0 < \mu_1 < \lambda_1 < \gamma_1 < \lambda_2 < \cdots \qquad (24)$$

hold. Then there exists, and moreover only one, Hill operator H_+ with potential $q(x)$, $q^{(m)}(x) \in L_1(0,a)$, of period a, whose spectrum coincides with the set Σ

$$\Sigma = \bigcup_{k=0}^{\infty} [\lambda_{2k}, \mu_{2k}] + \bigcup_{k=0}^{\infty} [\mu_{2k+1}, \lambda_{2k+1}] + \bigcup_{k=0}^{\infty} \gamma_k.$$

The numbers γ_k are eigenvalues of the operator H_+.

Theorem 6. Suppose that two sequences of real numbers $\{\lambda_\nu\}$, $\{\mu_\nu\}$ are given, satisfying inequalities (5), and suppose that the intervals Z_j of the real line are defined by formula (6). Assume that on the set $\bigcup_{j=0}^{\infty} Z_j$ a real function $p(\lambda)$ is given. In order that there exist a Hill operator with potential $q(x)$ of period a, where $q^{(m)}(x) \in L_1(0,a)$, and with spectral matrix $\rho(\lambda) = \|\rho_{ij}(\lambda)\|_{i,j}^{1,2}$ such that $\rho_{11}(\lambda) = p(\lambda)$, it is necessary and sufficient that $p(\lambda)$ admit a factorization of the form $p(\lambda) = \varphi(\lambda)/(D(\lambda))^{1/2}$, where $\varphi(\lambda)$ and $D(\lambda)$ are entire functions satisfying the conditions of Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.

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